An autoregressive (AR) model based stochastic unknown input realization and filtering technique

Dan Yu, Suman Chakravorty

Abstract— **This paper studies the state estimation problem of linear discrete-time systems with stochastic unknown inputs. The unknown input is a wide-sense stationary process while no other prior informaton needs to be known. We propose an autoregressive (AR) model based unknown input realization technique which allows us to recover the input statistics from the output data by solving an appropriate least squares problem, then fit an AR model to the recovered input statistics and construct an innovations model of the unknown inputs using the eigensystem realization algorithm (ERA). An augmented state system is constructed and the standard Kalman filter is applied for state estimation. A reduced order model (ROM) filter is also introduced to reduce the computational cost of the Kalman filter. One numerical example is given to illustrate the procedure.**

I. Introduction

In this paper, we consider the state estimation problem for systems with unknown stochastic inputs. The main contribution of our work is that when no prior information of the unknown inputs is known, we recover the statistics of the unknown inputs from the measurements, and then construct an innovations model of the unknown inputs from the recovered statistics such that the standard Kalman filter can be applied for state estimation. The innovations model is constructed by fitting an autoregressive (AR) model to the recovered input correlation data from which a state space model is constructed using the balanced realization technique. The method is tested on stochastically perturbed laminar flow problem.

The problem of state estimation of systems with unknown inputs has received considerable attention over the past few decades. The unknown input observer (UIO) has been well established for deterministic systems [1]–[3]. Various methods of building full-order or reduced-order observers have been developed, such as [4]–[6]. Recently, sliding mode observers have been proposed for systems with unknown inputs [7]. The design parameters and matrices need to be well chosen to satisfy certain conditions in order for the observers to perform well. For systems without the "observer matching" condition being satisfied, a high-gain approach is proposed [8]. The high-gain observers are used as approximate differentiators to obtain the estimates of the auxiliary outputs. In the presence of measurement noise, the high-gain observer amplifies the noise, and extra care needs to be taken when designing the gain matrix.

D. Yu is a Graduate Student Researcher, Department of Aerospace Engineering, Texas A&M University, College Station

S. Chakravorty is an Associate Professor of Aerospace Engineering, Texas A&M University, College Station

For stochastic systems, the problem of state estimation is known as unknown input filtering (UIF), and many UIF approaches are based on the Kalman filter [9]. When the dynamics of the unknown inputs is available, for example, if it can be assumed to be a wide-sense stationary process with known mean and covariance, one common approach called Augmented State Kalman Filter (ASKF) is used, where the states are augmented with the unknown inputs [10]. To reduce the computational complexity of ASKF, optimal two-stage and three-stage Kalman filters have been developed to decouple the augmented filter into two parallel reduced-order filters by applying a U-V transformation [11], [12]. When no prior information about the unknown input is available, an unbiased minimum-variance (UMV) filtering technique has been developed [13], [14]. The problem is transformed into finding a gain matrix such that the trace of the estimation error matrix is minimized. Certain algebraic constraints must be satisfied for the unbiased estimator to exist. In both the approaches above, the process noise is assumed to be white noise with known covariance.

In practice, there are many applications where the unknown inputs can be modeled as a stochastic process. For example, the state estimation of perturbed laminar flows is considered in [15]. It shows that the external disturbances (as well as the sensor noise and initial conditions) can be modeled as unknown stochastic inputs which perturb the linearized Navier-Stoke equations. Thus, the state estimation problem of such system is transformed into the unknown input filtering problem with stochastic unknown inputs. Also, our work can be applied to identify the statistics of colored process noise. There is some research that considers the Kalman filtering with unknown noise covariances [16]. The process noise is assumed to be white noise with unknown covariance, while in our approach, the process noise can be colored in time as well. There are also applications of our technique in signal processing, such as the wideband power spectrum estimation [17], where the problem is to recover the unknown power spectrum of a wide-sense stationary signal from the obtained sub-Nyquist rate samples.

In this paper, we address the state estimation problem of systems with stochastic unknown inputs. The unknown inputs are assumed to be wide sense stationary, while no other information about the unknown inputs is known. We propose a new unknown input filtering approach based on system realization techniques. Instead of constructing the gain matrix which needs to satisfy certain constraints, we apply the standard Kalman filtering using the following procedure: 1) recover the statistics of the unknown inputs from the measurements by solving an appropriate least squares problem, 2) find a spectral factorization of unknown input process by fitting an autoregressive (AR) model, 3) construct an innovations model of the unknown inputs via the eigensystem realization algorithm (ERA) [18] to the recovered input correlation data, and 4) apply the Augmented State Kalman Filter for state estimation. Different from existing methods, we construct a stochastic unknown input model from sensor data, which can be colored in time. To reduce the computational cost of the ASKF, we apply the Balanced Proper Orthogonal Decomposition (BPOD) technique [19] to construct a reduced order model (ROM) for filtering.

The paper is organized as follows. In Section II, the problem is formulated, and general assumptions are made about the system and the unknown inputs. In Section III, the AR based unknown input realization approach is proposed. The unknown input statistics are recovered from the measurements, then a linear model is constructed using an AR model and the ERA is used to generate a balanced minimal realization of the unknown inputs. After an innovations model of the unknown inputs is constructed, the ASKF is applied for state estimation in Section IV. Also, a ROM constructed using the BPOD is introduced to reduce the computational cost of Kalman filter. Section V presents one numerical example that utilize the proposed technique.

II. Problem Formulation

Consider a complex valued linear time-invariant discrete time system:

$$x_k = Ax_{k-1} + Bu_{k-1}, y_k = Cx_k + v_k, \tag{1}$$

where $x_k \in \mathbb{C}^n$, $y_k \in \mathbb{C}^q$, $v_k \in \mathbb{C}^q$, $u_k \in \mathbb{C}^p$ are the state vector, the measurement vector, the measurement white noise with known covariance, and the unknown stochastic inputs respectively. The process u_k is used to model the presence of the external disturbances, process noise, and unmodelled terms. Here, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{q\times n}$ are known.

The following assumptions are made about the system (1):

A1. A is a stable matrix.

A2. rank$(B) = p$, rank$(C) = q$, rank(CB)= rank(B) which implies that $p \leq q$.

A3. u_k and v_k are uncorrelated.

A4. We further assume that the unknown input u_k is generated by a linear stochastic system:

$$\xi_k = A_e\xi_{k-1} + B_e\nu_{k-1}, u_k = C_e\xi_k + \mu_k, \tag{2}$$

where ν_k, μ_k are uncorrelated white noise processes.

Remark 1: A2 is a general assumption in unknown input observer/filtering, the so-called "observer matching" condition. A4 implies that u_k is a wide-sense stationary(WSS) process with a rational power spectrum.

In this paper, we consider the state estimation problem when the system (2), i.e., (A_e, B_e, C_e) are unknown. Given the output data y_k, we want to construct an innovations model for the unknown stochastic input u_k, such that the output statistics of the innovations model and system (2) are the same. Given such a realization of the unknown input, we apply the standard Kalman filter for state estimation, augmented with the unknown input states.

III. AR based Unknown Input Realization Technique

In this section, we propose an AR based unknown input realization technique which can construct an innovations model of the unknown inputs such that the ASKF can be applied for state estimation. First, a least squares problem is formulated based on the relationship between the inputs and outputs to recover the statistics of the unknown inputs. Then an AR model is constructed using the recovered input statistics, and a balanced realization model is constructed using the ERA.

A. Extraction of Input Autocorrelations via a Least Squares Problem

Consider system (1) with zero initial conditions, the output y_k can be written as:

$$y_k = \sum_{i=1}^{\infty} h_i u_{k-i} + v_k, \tag{3}$$

where $h_i = CA^{i-1}B, i = 1, 2, \cdots$ are the Markov parameters of system (1).

For a linear time-invariant (LTI) system, under assumption A1, the output $\{y_k\}$ is a WSS process when $\{u_k\}$ is WSS. From the definition of the autocorrelation function of a WSS process, the output autocorrelation can be written as:

$$\begin{aligned} R_{yy}(m) &= E[y_k y_{k+m}^*] \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_i R_{uu}(m+i-j)h_j^* + R_{vv}(m), \end{aligned} \tag{4}$$

where $m = 0, \pm1, \pm2, \cdots$ is the time-lag between y_k and y_{k+m}. Here, assumption A3 is used. We use x^* to denote the complex conjugate transpose of x, and x^T to denote the transpose of x.

Since $v_k \sim N(0, \Omega)$, $R_{vv}(m) = \Omega$ for $m = 0$, and $R_{vv}(m) = 0$, otherwise. We denote $\hat{R}_{yy}(m) = R_{yy}(m) - R_{vv}(m)$, and hence, the relationship between input and output autocorrelation function is given by:

$$\hat{R}_{yy}(m) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_i R_{uu}(m+i-j)h_j^*. \tag{5}$$

For multiple input multiple output (MIMO) systems, h_i, $\hat{R}_{yy}(m)$, $R_{uu}(m)$ are matrices. To solve for the unknown input autocorrelations $R_{uu}(m)$, first we need to use a theorem from linear matrix equations [20].

Theorem 1: Consider the matrix equation

$$AXB = C, \tag{6}$$

where A, B, C, X are all matrices. If $A \in \mathbb{C}^{m\times n} = (a_1, a_2, \cdots, a_n)$, where a_i are the columns of A, then define vec$(A) \in \mathbb{C}^{mn\times 1}$ as: vec$(A) = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{pmatrix}^*$.

The matrix equation (6) can be transformed into one vector equation:

$$(B^T \otimes A)\text{vec}(X) = \text{vec}(C), \tag{7}$$

where $B^T \otimes A$ is the Kronecker product of B^T and A. If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}. \tag{8}$$

By applying Theorem 1, (5) can be written as:

$$\underbrace{\text{vec}(\hat{R}_{yy}(m))}_{\in R^{q^2 \times 1}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \underbrace{\bar{h}_j \otimes h_i}_{\in R^{q^2 \times p^2}} \underbrace{\text{vec}(R_{uu}(m+i-j))}_{\in R^{p^2 \times 1}}, \tag{9}$$

where $\bar{h}_i$ denotes the matrix h_i with complex conjugated entries, and $h_i^* = (\bar{h}_i)^T$. Now, we estimate the unknown input autocorrelations by the following procedure.

1) Choose design parameter M: For a stable system, we make the following assumption.

A5. Assume that there exists a finite number M such that the Markov parameters of the system $\|h_i\| \leq \delta, i > M$, where δ is small enough.

Here, $\|A\|$ denotes the Frobenius norm of matrix A, and $\|x\|_2$ denotes the Euclidean norm of vector x. M is a design parameter that varies with different systems and can be chosen as large as desired. Thus, (9) can be written as:

$$\text{vec}(\hat{R}_{yy}(m)) = \sum_{i=1}^{M}\sum_{j=1}^{M} \bar{h}_j \otimes h_i \text{vec}(R_{uu}(m+i-j)). \tag{10}$$

2) Choose design parameters N_o, N_i: If $\{u_k\}$ is WSS, then we make the following assumption.

A6. $\hat{R}_{yy}(m)$ only has significant values within a range $-N_o \leq m \leq N_o$, and negligible values outside this range. Also, we assume the support of $R_{uu}(m)$ is limited to $-N_i \leq m \leq N_i$.

This is a rather standard assumption when computing a power spectrum from an autocorrelation function. The numbers N_o and N_i depend on the dynamic system and unknown inputs, and are design parameters that can be chosen as large as required.

Under assumption A6, we have the following proposition.

Proposition 1: The relation $N_i \leq N_o$ holds, which implies that all significant input autocorrelations can be recovered from the output autocorrelations.

Proof: From the assumption that the support of $\hat{R}_{yy}$ is limited to $[-N_o, N_o]$, we have: $\hat{R}_{yy}(N_o+1) = 0$.

Using (9), $\text{vec}(\hat{R}_{yy}(N_o+1)) = \sum_{i=1}^{\infty} \bar{h}_i \otimes h_i \text{vec}(R_{uu}(N_o+1)) + \sum_{i=2}^{\infty} \bar{h}_{i-1} \otimes h_i \text{vec}(R_{uu}(N_o)) + \cdots$.

If $N_i > N_o$, which means $R_{uu}(N_o+1) \neq 0$, then it follows that $R_{yy}(N_o+1)$ is also not negligible, which contradicts the assumption. Thus, as a consequence, $N_i \leq N_o$. ■

The following equation is used for computation of the unknown input autocorrelations.

$$\text{vec}(\hat{R}_{yy}(m)) = \sum_{i=1}^{M}\sum_{j=1}^{M} \bar{h}_j \otimes h_i \text{vec}(R_{uu}(m+i-j)), \tag{11}$$

where $|m| \leq N_o$, and $|m+i-j| \leq N_i$.

3) Solve the least squares problem: We collect $2N_o+1$ output autocorrelations, and there are $2N_i+1$ unknown input autocorrelations:

$$\underbrace{\begin{pmatrix} \text{vec}(\hat{R}_{yy}(-N_o)) \\ \vdots \\ \text{vec}(\hat{R}_{yy}(0)) \\ \vdots \\ \text{vec}(\hat{R}_{yy}(N_o)) \end{pmatrix}}_{\text{vec}(\hat{R}_{yy})} = C_{yu} \underbrace{\begin{pmatrix} \text{vec}(R_{uu}(-N_i)) \\ \vdots \\ \text{vec}(R_{uu}(0)) \\ \vdots \\ \text{vec}(R_{uu}(N_i)) \end{pmatrix}}_{\text{vec}(R_{uu})}, \tag{12}$$

where C_{yu} is the coefficient matrix and can be calculated from (11).

Under assumption A2 and A6, we have the following proposition.

Proposition 2: Equation (12) has a unique least squares solution $\hat{R}_{uu}(m), m = \pm1, \pm2, \cdots, \pm N_i$.

Proof: First, we prove that A2 is a necessary condition. The (i^{th}, j^{th}) term in C_{yu} is:

$$C_{yu}(i,j) = \sum_s \sum_t \bar{h}_t \otimes h_s, \text{for some } s, t \tag{13}$$

where $i = 1, 2, \cdots, 2N_o+1, j = 1, 2, \cdots, 2N_i+1$.

Since $h_i = CA^{i-1}B \in \mathbb{C}^{q \times p}$, $\text{rank}(h_i) \leq \min(p,q) = k$. From the property that $\text{rank}(A \otimes B) = \text{rank}(A)\text{rank}(B)$, we have $\text{rank}(\bar{h}_t \otimes h_s)$ = $\text{rank}(\bar{h}_t)\text{rank}(h_s) = k^2$, which means $C_{yu}(i,j)$ has at most k^2 independent columns. It follows that C_{yu} has at most $k^2(2N_i+1)$ independent columns, i.e., $\text{rank}(C_{yu}) \leq k^2(2N_i+1)$. For C_{yu} to have a left inverse, C_{yu} should have full column rank, i.e., $\text{rank}(C_{yu}) = p^2(2N_i+1)$. Thus, C_{yu} has a left inverse if and only if $k = p$, which implies $\min(p,q) = p$, and hence, $p \leq q$.

There are $q^2(2N_o+1)$ equations with $p^2(2N_i+1)$ unknowns, from the assumptions $p \leq q$ and $N_i \leq N_o$, there is an unique solution $\hat{R}_{uu}(m), m = 0, \pm1, \cdots, \pm N_i$ in the least square sense, and $\hat{R}_{uu}(m)$ is the input autocorrelations we extract from the output autocorrelations. ■

Remark 2: The size of C_{yu} is $q^2(2N_o+1) \times p^2(2N_i+1)$ and it would be large when p and q increase, and hence, large scale least squares problem needs to be solved for systems with large number of inputs/outputs. For example, a modified conjugate gradients method [21] could be used as follows.

The least squares problem need to be solved is:

$$\text{vec}(\hat{R}_{yy}) = C_{yu}\text{vec}(R_{uu}). \tag{14}$$

Multiply C_{yu}^* on both sides, denote $L_s = C_{yu}^*\text{vec}(\hat{R}_{yy})$, $\bar{x} = \text{vec}(R_{uu})$, and $C_s = C_{yu}^* C_{yu}$, thus $C_s = C_s^*$, and the problem is equivalent to solve the least squares problem $C_s\bar{x} = L_s$ for $\bar{x}$. A conjugate gradient method to solve this problem is summarized in Algorithm 1.

Algorithm 1 Conjugate gradient algorithm

1) For a least squares problem $C_s\bar{x} = L_s$, where $C_s = C_s^*$, $\bar{x}$ is unknown.
2) Start with a randomly initial solution $\bar{x}_0$.
3) $r_0 = L_s - C_s\bar{x}_0$, $p_0 = r_0$.
4) for $k = 0$, repeat
5) $\alpha_k = \frac{r_k^* r_k}{p_k^* C_s p_k}$,
   $\bar{x}_{k+1} = \bar{x}_k + \alpha_k p_k$,
   $r_{k+1} = r_k - \alpha_k C_s p_k$,
   if r_{k+1} is sufficient small then exit loop.
   $\beta_k = \frac{r_{k+1}^* r_{k+1}}{r_k^* r_k}$,
   $p_{k+1} = r_{k+1} + \beta_k p_k$,
   $k = k + 1$,
   end repeat.
6) The optimal estimation is x_{k+1}.

Denote $R_{uu}(m)$ as the "true" input autocorrelations, and $\Delta(m) = R_{uu}(m) - \hat{R}_{uu}(m)$ as the error of the input autocorrelations we extract, $\Delta(m)$ results from two design parameters: the choice of M and N_i. We analyze the errors seperately, in the following.

Proposition 3: Denote $R_{uu}^M(m)$ as the input autocorrelations we extract by using M Markov parameters of the dynamic system. The errors of input autocorrelations resulting from assumption A5 is: $\|\Delta_M(m)\| \le k_M\delta$, where k_M is some constant, δ is defined in assumption A5.

The Perturbation theory [22] is used to prove the above result, and the proof is shown in Appendix .

Remark 3: Error analysis in the Fourier domain.

The power spetral density is defined as:

$$S_{uu}(\omega) = \sum_{k=-\infty}^{\infty} R_{uu}(k)e^{-jk\omega}, \tag{15}$$

$$S_{yy}(\omega) = \sum_{k=-\infty}^{\infty} \hat{R}_{yy}(k)e^{-jk\omega}, \tag{16}$$

Thus, by substituting (5), the relationship between the output power spectral density and input power spectral density is:

$$\begin{aligned} S_{yy}(\omega) &= \sum_{k=-\infty}^{\infty} R_{uu}^M(k)e^{-jk\omega} + \Delta S_M(\omega) \\ &= S_{yy}^M(\omega) + \Delta S_M(\omega), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Delta S_M(\omega) &= \sum_{k=-\infty}^{\infty} h_{M+1}R_{uu}(k)h_{M+1}^* e^{-jk\omega} \\ &+ \sum_{k=-\infty}^{\infty} h_{M+1}R_{uu}(k)h_1^* e^{-j(k-M)\omega} + \cdots \\ &= h_{M+1}S_{uu}(\omega)h_{M+1}^* + h_{M+1}S_{uu}(\omega)e^{jM\omega}h_1^* + \cdots . \end{aligned} \tag{18}$$

Thus, $\|\Delta S_M(\omega)\| \le k_1\delta$, where k_1 is some constant. Hence, the truncation error by using M Markov parameters can be seen to be a small perturbation in the frequency domain.

Proposition 4: Denote $R_{uu}^N(m)$ as the input autocorrelations we extract under assumption A6. The errors resulting from this assumption is $\|\Delta_N(m)\| \le k_N\delta$, where k_N is some constant, δ is defined in assumption A6.

The proof is similarly like the proof of Proposition 3, and is omitted here due to the page limit.

Remark 4: Error analysis in frequency domain:

$$S_{yy}(\omega) = S_{yy}^N(\omega) + \Delta S_N(\omega), \tag{19}$$

where $S_{yy}^N(\omega) = \sum_{k=-\infty}^{\infty} R_{uu}^N(k)e^{-jk\omega}$, and

$$\|\Delta S_N(\omega)\| \le \sum_{k=-\infty}^{\infty} (\sum_{i=1}^{\infty}\sum_{t=1}^{\infty} \delta\|h_i\|\|h_t^*\|)e^{-jk\omega}\| \le k_2\delta, \tag{20}$$

where k_2 is some constant.

Under the assumptions A5 and A6, the following proposition considers the total errors of input autocorrelations we recover.

Proposition 5: Denote $\hat{R}_{uu}(m)$ as the input autocorrelation function we estimate from the output autocorrelations, and let $\Delta(m) = R_{uu}(m) - \hat{R}_{uu}(m)$ be the error between the estimated input autocorrelation and the "true" input autocorrleation. Then $\|\Delta(m)\| \le k\delta$, where k is some constant.

Proposition 3 and 4 are used for the proof, and is omitted here due to the page limit. The results above show that if M, N_i, N_o are chosen large enough, the errors in estimating the input autocorrelations can be made arbitrarily small.

B. Construction of the AR Based Innovations Model

After we extract the input autocorrelations from the output autocorrelations, we want to construct a system which will generate the same statistics as the ones we recovered in Section III-A. If assumption A4 is satisfied, the power spectrum of u_k is continuous, and can be factored [23]. Such system can be constructed by using an Autoregressive(AR) model. In an AR model, the time series can be expressed as a linear function of its past values, i.e.,

$$u(k) = \sum_{i=1}^{M_i} a_i u(k-i) + \epsilon(k), \tag{21}$$

where $\epsilon(k)$ is white noise with distribution $N(0, \Omega_r)$, M_i is the order of the AR model, and $a_i, i = 1, 2, \cdots, M_i$ are the coefficient matrices. For a vector autoregressive model with complex values, the Yule-Walker equation [24] which is used to solve for the coefficients needs to be modified. The modified Yule-Walker equation can be written as:

$$\begin{pmatrix} R_{uu}(-1) & R_{uu}(-2) & \cdots & R_{uu}(-M_i) \end{pmatrix} = \begin{pmatrix} a_1^* \\ \vdots \\ a_{M_i}^* \end{pmatrix}^* \times \begin{pmatrix} R_{uu}(0) & R_{uu}(-1) & \cdots & R_{uu}(1-M_i) \\ R_{uu}(1) & R_{uu}(0) & \cdots & R_{uu}(2-M_i) \\ \vdots & \vdots & \vdots & \vdots \\ R_{uu}(M_i-1) & R_{uu}(M_i-2) & \cdots & R_{uu}(0) \end{pmatrix}. \tag{22}$$

Equation (22) is used to solve for the coefficient matrices $a_i, i = 1, 2, \cdots, M_i$. The covariance of the residual white noise $\epsilon(k)$ can be solved using the following equation:

$$R_{\epsilon\epsilon}(m) = R_{uu}(m) - \sum_{i=1}^{M_i}\sum_{j=1}^{M_i} a_i R_{uu}(m+i-j)a_j^*, \quad (23)$$

where $\Omega_r = R_{\epsilon\epsilon}(0)$. The balanced minimal realization for the AR model (21) can be expressed as:

$$\eta_k = A_n\eta_{k-1} + B_n u_{k-1}, u_k = C_n\eta_k + \epsilon_k, \quad (24)$$

where (A_n, B_n, C_n) are solved by using the ERA technique [18] with $a_i, i = 1, \cdots, M_i$ as the Markov parameters of the system. Equation (24) is equivalent to:

$$\eta_k = (A_n + B_nC_n)\eta_{k-1} + B_n\epsilon_{k-1}, u_k = C_n\eta_k + \epsilon_k, \quad (25)$$

where ϵ_k is white noise with covariance Ω_r. By using the Cholesky Decomposition, we can find a unique lower triangular matrix P such that: $\Omega_r = PP^*$.

If w_k is white noise with distribution $N(0, 1)$, then Pw_k would be white noise with distribution $N(0, \Omega_r)$. Thus, the innovation model we construct that has the same statistics as the unknown input system (2) is:

$$\eta_k = (A_n + B_nC_n)\eta_{k-1} + B_nPw_{k-1}, u_k = C_n\eta_k + Pw_k, \quad (26)$$

where w_k is a randomly white noise with standard normal distribution.

Under assumption A4, we have the following proposition.

Proposition 6: Denote $\hat{R}_{uu}(m)$ as the input autocorrelations recovered from the measurements, then $\hat{R}_{uu}(m)$ can be reconstructed exactly by using the innovations model (26), i.e., $\tilde{R}_{uu}(m) = \hat{R}_{uu}(m)$, where $\tilde{R}_{uu}(m)$ is the input autocorrelations of the realization of system (26).

From Proposition 5 and 6, under the same assumptions, the following corollary immediately follows.

Corollary 1: Denote u_k as the actual unknown input process, and $R_{uu}(m)$ as the actual input autocorrelation function. Then $\|\tilde{R}_{uu}(m) - R_{uu}(m)\| \leq k_a\delta$, where k_a is some constant, when δ is small enough. System (26) is an innovations model for the unknown input u_k.

The procedure of constructing the innovations model is summarized in Algorithm 2.

Remark 5: For real valued system, we can save the computation by using the properities of autocorrelation functions:

$$R_{u_iu_i}(-m) = R_{u_iu_i}(m), R_{u_iu_j}(-m) = R_{u_ju_i}(m), i \neq j \quad (27)$$

Thus, we only need to collect $N_o + 1$ output autocorrelations and have $p^2(N_o + 1)$ equations with $q^2(N_i + 1)$ unknowns in (12).

IV. Augmented State Kalman Filter and Model Reduction

After we construct an innovations model for the unknown inputs, we apply the standard Kalman filter on the augmented system with states augmented by the unknown input states. A ROM based filter is also constructed using the BPOD for reducing the computational cost of the resulting filter.

Algorithm 2 AR based unknown input realization technique

1) Choose finite number N_o, compute output autocorrelation function $R_{yy}(m)$ by using measurements y_k, $|m| \leq N_o$.
2) Choose finite number M, construct the coefficient matrix C_{yu} from (11).
3) Choose finite number N_i, solve the least squares problem (12) for unknown input autocorrelation function $R_{uu}(m)$, $|m| \leq N_i$.
4) Construct an AR model for the unknown input $u(k) = \sum_{i=1}^{M_i} a_i u(k-i) + \epsilon(k)$, find the coefficient matrices $a_i, i = 1, 2, \cdots M_i$ by solving the modified Yule-Walker equation (22).
5) Find the covariance Ω_r of $\epsilon(k)$ by solving (23).
6) Construct the state space representation (24) for the AR model using ERA.
7) Find a unique lower triangular matrix P such that $\Omega_r = PP^*$, and construct an innovations model as in (26).

A. Augmented State Kalman Filter

The full order system can be represented by augmenting the states of the original system as:

$$\begin{pmatrix} x_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} A & BC_n \\ 0 & A_n + B_nC_n \end{pmatrix}\begin{pmatrix} x_k \\ \eta_k \end{pmatrix} + \begin{pmatrix} BP \\ B_nP \end{pmatrix} w_k,$$
$$y_k = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x_k \\ \eta_k \end{pmatrix} + v_k, \quad (28)$$

where w_k is white noise with standard normal distribution. v_k is white noise with known covariance. Thus, we may now use the standard kalman filter for state estimation of the augmented system (28).

B. Unknown Input Estimation Using Model Reduction

For large scale systems, we can use model reduction technique such as Balanced Proper Orthogonal Decomposition (BPOD) [19] to construct a reduced order model (ROM) first, and then extract the input autocorrelations from the reduced order model. We apply the Kalman filter to the ROM to reduce the computational cost. For a large scale system with a large number of inputs and outputs, we can also use the randomized proper orthogonal decomposition (RPOD) technique [25] for model reduction.

The ROM system is extracted from the full order system using the BPOD and is denoted by:

$$x_k = A_r x_{k-1} + B_r u_{k-1},$$
$$y_k = C_r x_k + v_k. \quad (29)$$

Let $\hat{h}_i = C_rA_r^{i-1}B_r, i = 1, 2, \cdots, M$ be the Markov parameters of the ROM. Then the relationship between input autocorrelations and output autocorrelations can be written as: $\hat{R}_{yy}(m) = \sum_{i=1}^{M}\sum_{j=1}^{M} \hat{h}_i R_{uu}(m+i-j)\hat{h}_j^*$.

Following the same procedure as in Algorithm 2, we can now recover the input autocorrelations, and construct an innovations model which can generate the same statistics as the unknown inputs. The advantage of using model reduction is that for a large scale system, computing $\hat{h}_i = C_r A_r^{i-1} B_r$ is much faster than computing $h_i = CA^{i-1}B$ because of the reduction in the size of A. Also, the order of the ROM is much smaller than the order of the full order system, and thus the computational cost of using the Kalman filter is much reduced. Hence, even with the augmented states, the standard Kalman filter remains computationally tractable.

Remark 6: To reduce the computational cost of the augmented states in Kalman filter, we can also use the existing optimal two-stage or three-stage kalman filtering technique [11], [12], which decouple the augmented filter into two parallel reduced order filters. These techniques are preferable when the order of the innovations model is high, while the BPOD based ROM filter is preferable when the order of the dynamic system is high.

V. Computational Results

We test the method on the perturbed laminar flow equation. We construct the unknown input system by using both the full order system as well as the ROM constructed by BPOD. We check the results by comparing the autocorrelation functions of the inputs, outputs and the states. Also, we show the state estimation using the Kalman filter.

A. Orr-Sommerfeld Equation

Consider the three-dimensional flow between two infinite plates (at $y = \pm 1$) driven by a gradient in the streamwise x direction. The mean velocity profile is given by $U(y) = 1 - y^2$. At each wavenumber pair $(\alpha, \beta)_{mn}$, the wall-normal velocity $v(x, y, z, t)$ and wall-normal vorticity $\eta(x, y, z, t)$ are:

$$v(x, y, z, t) = \hat{v}_{mn}(y, t) e^{i(\alpha x + \beta z)},$$
$$\eta(x, y, z, t) = \hat{\eta}_{mn}(y, t) e^{i(\alpha x + \beta z)}. \quad (30)$$

Denote $\hat{q}_{mn}(y, t) = \begin{pmatrix} \hat{v}_{mn}(y, t) \\ \hat{\eta}_{mn}(y, t) \end{pmatrix}$, where $\hat{(.)}$ denotes the Fourier transformed variable, and $(.)_{mn}$ denotes the wavenumber pair $(\alpha, \beta)_{mn}$. The evolution of the flow in Fourier domain can be written as:

$$\frac{d}{dt} M \hat{q}_{mn} + L \hat{q}_{mn} = T f(y, t), \quad (31)$$

where

$$M = \begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix}, \quad (32)$$

$$L = \begin{pmatrix} -i\alpha U \Delta + i\alpha U'' + \Delta^2/Re & 0 \\ i\beta U' & i\alpha U - \Delta/Re \end{pmatrix}. \quad (33)$$

Operater T transforms the forcing $f = (f_1, f_2, f_3)^T$ on the evolution equation for the velocity vector $(u, v, w)^T$ into an equivalent forcing on the $(v, \eta)^T$ system [15],

$$T = \begin{pmatrix} i\alpha D & k^2 & i\beta D \\ i\beta & 0 & -i\alpha \end{pmatrix}, k^2 = \alpha^2 + \beta^2, \Delta = D^2 - k^2, \quad (34)$$

and D, D^2 represent the first and second order differentiation operators in the wall-normal direction. The forcing $f(y, t)$ accounts for the nonlinear terms and the external disturbances via an unknown stochastic model.

The boundary conditions on v and η correspond to no-slip solid walls $v(\pm 1) = Dv(\pm 1) = \eta(\pm 1) = 0$.

System (31) can be discretized using Chebyshev polynomials, and in the simulation, we assume there are two unknown inputs and two measurements.

In the simulation, the design parameters $M = 1000$, $N_i = N_o = 100$ are chosen as follows. M is chosen so that the Markov parameters $\|h_i\| \approx 0, i > M$. N_i and N_o are chosen by trial and error. First, we randomly choose a suitable N_i and N_o, where $N_i \leq N_o$. Then we follow the AR based unknown input realization procedure, and construct the augmented state system (28). Given the white noise processes w_k, v_k perturbing the system, we check the output statistics of the augmented state system (28). If the errors are small enough, we stop, otherwise, we increase the values of N_i and N_o, and repeat the same procedure until the errors are negligible. Notice that increasing M, N_i, N_o would increase the accuracy of the input statistics we can recover, but also increases the computational cost. The unknown input f is assumed to be a colored noise generated by a third order linear complex system. The realization of the unknown inputs is a second order system. The measurement noise is white noise with covariance $0.1 I_{2\times 2}$.

First, we show the comparison of the input autocorrelations we recover with the actual input autocorrelations in complex plane in Fig. 1. Since there are two inputs, thus, the cross-correlation function between input 1 and input 2 are also included in the input autocorrelations.

Before we apply the ASKF for the state estimation, we compare the statistics of the states and outputs of the system perturbed by the unknown inputs we construct and the actual system. Fig. 2 shows the comparison between the estimated output autocorrelations and the actual autocorrelations. The comparison of the state autocorrelations is omitted here due to the page limit.

It can be seen that the statistics of the unknown inputs can be recovered almost perfectly, and given the system perturbed by the unknown inputs innovations model we constructed,

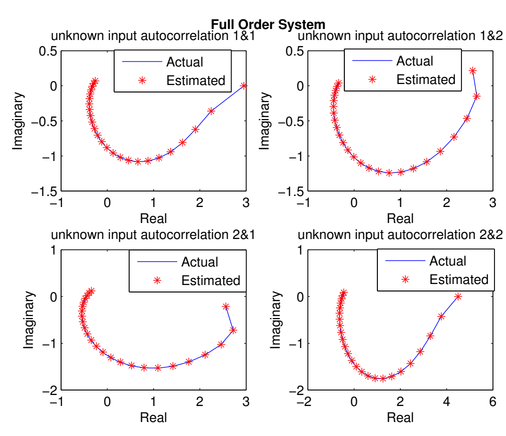

Fig. 1. Comparison of input autocorrelations

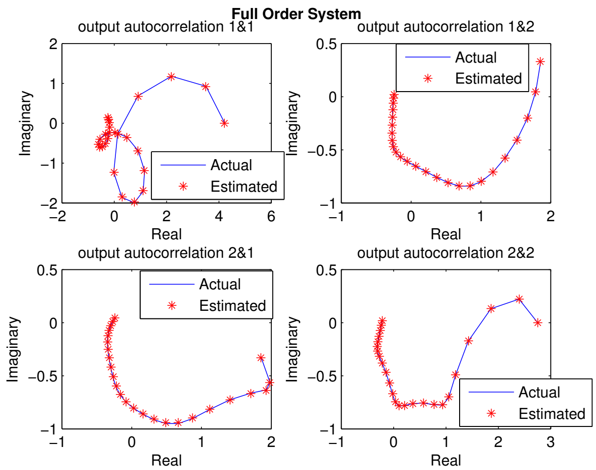

Fig. 2. Comparison of output autocorrelations

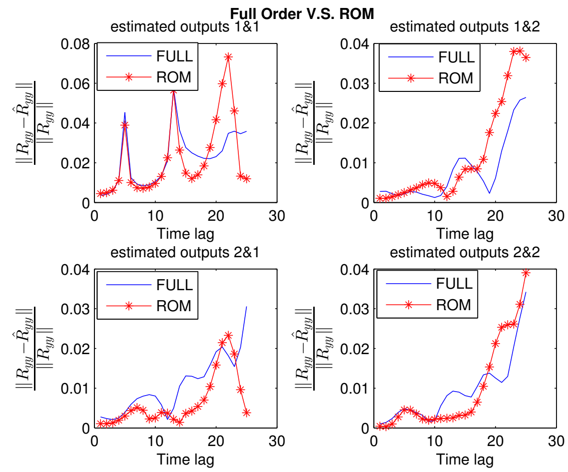

Fig. 4. Comparison of output autocorrelation relative error

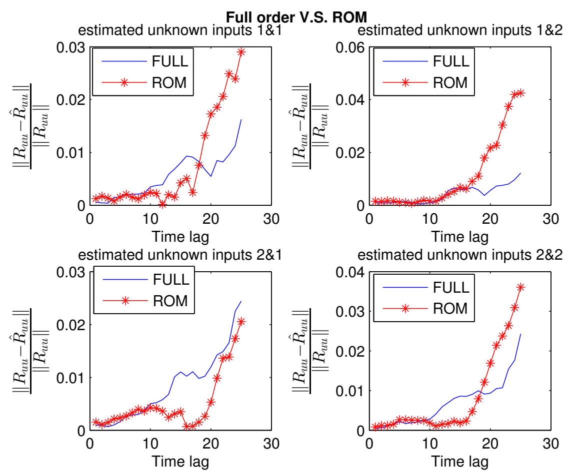

Fig. 3. Comparison of input autocorrelation relative error

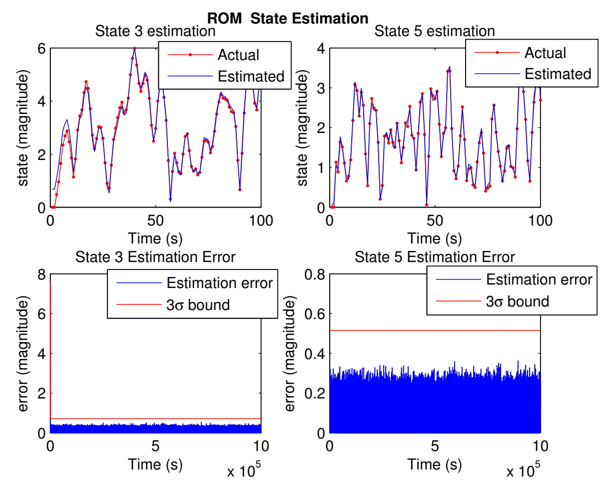

Fig. 5. State estimation using ROM

the statistics of the outputs are almost the same as well.

Next, we compare the performance of the unknown inputs constructed using the ROM with the full order system. The full order system has 30 states, and the ROM has 15 states. The relative error of the input autocorrelation is shown in Fig. 3, and the comparison of the relative error of output autocorrelations is shown in Fig.4.

We can see that the statistics reconstructed by using the ROM is not as accurate as using the full order system, however, the relative error is on the same scale, and hence, the computational cost is reduced without losing too much accuracy.

The comparison of the state estimation using the ROM is shown in Fig. 5. The behavior of the ASKF using full order system is similar, and is omitted here. We randomly choose two states and show the comparison of the acutal state with the estimated states. The state estimation error and 3σ bounds are shown. Since the error is complex valued, only the absolute value of the error is shown. It can be seen that the kalman filter using the ROM perform well, and hence, for a large scale system, the computational complexity of ASKF can be reduced by using the BPOD.

VI. Conclusion

In this paper, we have proposed a balanced unknown input realization method for the state estimation of system with unknown stochastic inputs. The unknown inputs are assumed to be a wide sense stationary process with a rational power spectrum, and no other prior information about the unknown inputs needs to be known. We recover the unknown inputs statistics from the output data using a least-squares procedure, and then construct a balanced minimal realization of the unknown inputs using an AR model and the ERA technique. The recovered innovations model is used for state estimation, and the standard Kalman filter is applied on the augmented system. The next step in this process would require us to consider more complex realistic problems in fluid flow application, and cases where the unknown numbers of inputs/ outputs are large, and also cases where the locations of the inputs are unknown.

APPENDIX
PROOF OF PROPOSITION 3

Proof: The output autocorrelation function using the first M Markov parameters is:

$$\hat{R}_{yy}^{M}(m) = \sum_{i=1}^{M}\sum_{j=1}^{M} h_i R_{uu}(m+i-j)h_j^*. \tag{35}$$

Comparing with (5), the output autocorrelation errors resulting from using M Markov parameters is:

$$\Delta_1(m) = \sum_{i=M+1}^{\infty}\sum_{j=1}^{M} h_i R_{uu}(m+i-j)h_j^* + \sum_{i=M+1}^{\infty}\sum_{j=M+1}^{\infty} h_i R_{uu}(m+i-j)h_j^* + \sum_{i=1}^{M}\sum_{j=M+1}^{\infty} h_i R_{uu}(m+i-j)h_j^*. \tag{36}$$

From assumption A5, by choosing M large enough, we have $\|h_i\| \le \delta, i > M$, where δ is small enough, thus,

$$\|\Delta_1(m)\| \le \sum_{i=M+1}^{\infty}\sum_{j=1}^{M} \delta \times \|R_{uu}(m+i-j)\|\|h_j^*\| + \sum_{i=M+1}^{\infty}\sum_{j=M+1}^{\infty} \delta \times \|R_{uu}(m+i-j)\| \times \delta + \sum_{i=1}^{M}\sum_{j=M+1}^{\infty} \|h_i\|\|R_{uu}(m+i-j)\| \times \delta \le k_3\delta, \tag{37}$$

where k_3 is some constant.

Denote C_{yu} as the "true" coefficient matrix and C_{yu}^M as the coefficient matrix using M Markov papameters, we need to solve the least squares problem:

$$\text{vec}(\hat{R}_{yy}) = C_{yu}^M \text{vec}(R_{uu}^M). \tag{38}$$

where R_{uu}^M is the input autocorrelation we recover from using M Markov parameters, and $\text{vec}(\hat{R}_{yy})$ is defined in (12).

Since $\|\text{vec}(\hat{R}_{yy}(m)) - \text{vec}(\hat{R}_{yy}^M(m))\|_2 = \|\hat{R}_{yy}(m) - \hat{R}_{yy}^M(m)\| = \|\Delta_1(m)\| \le k_3\delta$, we have $\text{vec}(\hat{R}_{yy}(m)) = \text{vec}(\hat{R}_{yy}^M(m)) + \Delta_2(m)$, where $\|\Delta_2(m)\|_2 \le k_3\delta$, or equivalently

$$\text{vec}(\hat{R}_{yy}) = \text{vec}(\hat{R}_{yy}^M) + \Delta_2, \tag{39}$$

Consider (12), $\text{vec}(\hat{R}_{yy})$ and $\text{vec}(\hat{R}_{yy}^M)$ can be written as:

$$\begin{aligned}\text{vec}(\hat{R}_{yy}) &= C_{yu}\text{vec}(R_{uu}),\\ \text{vec}(\hat{R}_{yy}^M(m)) &= C_{yu}^M\text{vec}(R_{uu}),\end{aligned} \tag{40}$$

Substitute into (39), and since $(C_{yu}^M)^{-1}$ exists, we have:

$$\text{vec }(R_{uu}) - \text{vec}(R_{uu}^M) = (C_{yu}^M)^{-1}\Delta_2, \tag{41}$$

which means: $\|\text{vec}(R_{uu}) - \text{vec}(R_{uu}^M)\|_2 \le k_M\delta$, where k_M is some constant. Thus, we have $\|\Delta_M(m)\| \le k_M\delta$, where k_M is some constant. ∎

REFERENCES

[1] S.-H. Wang, E.J.Davison, and P. Dorato, "Observing the states of systems with unmeasurable disturbances," *IEEE Transactions on Automatic Control*, vol. 20,No.5, pp. 716–717, 1975.

[2] S. Bhattacharyya, "Observer design for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. AC-23,No.3, pp. 483–484, 1978.

[3] P. Kudva, N.Viswanadham, and A. Ramakrishna, "Observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 25,No.1, pp. 113–115, 1980.

[4] M. Hou and P. Muller, "Design of observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 37, No.6, pp. 871–875, 1992.

[5] S. Hui and S. H. Zak, "Low-order state estimators and compensators for dynamical systems with unknown inputs," *Systems & Control Letters*, vol. 21, No.6, pp. 493–502, 1993.

[6] M. Darouach, M. Zasadzinski, and S. Xu, "Full-order observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 39, No.3, pp. 606–609, 1994.

[7] S. K. Spurgeon, "Sliding mode observers: a survey," *International Journal of Systems Science*, vol. 39, No.8, pp. 751–764, 2008.

[8] K. Kalsi, J. Lian, S. Hui, and S. H. Zak, "Sliding-mode observers for systems with unknown inputs: A high-gain approach," *Automatica*, vol. 46, Issue 2, pp. 347–353, 2010.

[9] M. Darouach, M. Zasadzinski, A. B. Onana, and S. Nowakowski, "Kalman filtering with unknown inputs via optimal state estimation of singular systems," *International Journal of Systems Science*, vol. 26(10), pp. 2015–2028, 1995.

[10] C.-S. Hsieh, "A unified framework for state estimation of nonlinear stochastic systems with unknown inputs," in *Proceedings of 9th IEEE Asian Control Conference*, 2013.

[11] C.-S. Hsieh and F.-C. Chen, "Optimal solution of the two-stage kalman estimator," *IEEE Transactions on Automatic Control*, vol. 44, pp. 194–199, 1999.

[12] F. B. Hmida, K. Khemiri, J. Ragot, and M. Gossa, "Three-stage kalman filter for state and fault estimation of linear stochastic systems with unknown inputs," *Journal of the Franklin Institute*, vol. 349, pp. 2369–2388, 2012.

[13] S. Gillijns and B. D. Moor, "Unbiased minimum-variance input and state estimation for linear discrete-time systems," *Automatica*, vol. 43, pp. 111–116, 2007.

[14] C.-S. Hsieh, "Extension of unbiased minimum-variance input and state estimation for systems with unknown inputs," *Automatica*, vol. 45, pp. 2149–2153, 2009.

[15] J. Hepffner, M. Chevalier, T. R. Bewley, and D. S. Henningson, "State estimation in wall-bounded flow systems. part 1. perturbed laminar flows," *Journal of Fluid Mechanics*, vol. 534, pp. 263–294, 2005.

[16] J. Dunik and M. Simandl, "Estimation of state and measurement noise covariance matrices by multi-step prediction," in *Proceedings of the 17th IFAC World Congress*, 2008, pp. 3689–3694.

[17] D. D. Ariananda and G. Leus, "Compressive wideband power spectrum estimation," *IEEE Transactions on signal processing*, vol. 60,No.9, pp. 4775–4789, 2012.

[18] J.-N. Juang, *Applied System Identification*. Englewood Cliffs, NJ: Prentice Hall, 1994.

[19] C. W. Rowley, "Model reduction for fluids using balanced proper orthogonal decomposition," *International Journal of Bifurcation and Chaos*, vol. 15, pp. 997–1013, 2005.

[20] W. E. Roth, "On direct product matrices," *Bulletin of the American Mathematical Society*, vol. 40, pp. 461–468, 1934.

[21] V. Faber and T. Manteuffel, "Necessary and sufficient conditions for the existence of a conjugate gradient method," *SIAM Journal on Numerical Analysis*, vol. 21, pp. 352–362, 1984.

[22] T. Kato, *Perturbation Theory for Linear Operators*. New York: Springer-Verlag, 1995.

[23] E. Wong and B. Hajek, *Stochastic Processes in Engineering Systems*. New York: Springer-Verlag, 1985.

[24] B. Friedlander and B. Porat, "The modified yule-walker method of arma spectral estimation," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-20, No.2, pp. 158–173, 1984.

[25] D. Yu and S. Chakravorty, "A randomized proper orthogonal decomposition technique," arXiv: 1312.3976, Tech. Rep., 2013, submitted to Automatica.