A stochastic unknown input realization and filtering technique ⋆

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Abstract

This paper studies the state estimation problem of linear discrete-time systems with unknown inputs which can be treated as a wide-sense stationary process with rational power spectral density, while no other prior information needs to be known. We propose an autoregressive (AR) model based unknown input realization technique which allows us to recover the input statistics from the output data by solving an appropriate least squares problem, then fit an AR model to the recovered input statistics and construct an innovations model of the unknown inputs using the eigensystem realization algorithm. An augmented state system is constructed and the standard Kalman filter is applied for the state estimation. A reduced order model filter is also introduced to reduce the computational cost of the Kalman filter. A numerical example is given to illustrate the procedure.

Key words: Unknown input filtering; ARMA models; System identification; Statistical analysis

1 Introduction

In this paper, we consider the state estimation problem for systems with unknown inputs. The main contribution of our work is that when no prior information of the unknown inputs is known, we recover the statistics of the unknown inputs from the measurements, and then construct an innovations model of the unknown inputs from the recovered statistics such that the standard Kalman filter can be applied for the state estimation. The innovations model is constructed by fitting an autoregressive (AR) model to the recovered input correlation data from which a state space model is constructed using the balanced realization technique. The method is tested on the stochastically perturbed heat transfer problem.

For stochastic systems, the state estimation problem with unknown inputs is known as unknown input filtering (UIF) problem, and many UIF approaches are based on the Kalman filter [1, 6, 14]. When the dynamics of the unknown inputs is available, for example, if it can be assumed to be a wide-sense stationary (WSS) process with known mean and covariance, one common approach called Augmented State Kalman Filter (ASKF) is used, where the states are augmented with the unknown inputs [9]. To reduce the computational complexity of ASKF, optimal two-stage Kalman filters (OTSKF) and optimal three-stage Kalman filters have been developed to decouple the augmented filter into two parallel reduced-order filters by applying a U-V transformation [5,10,12]. When no prior information about the unknown input is available, an unbiased minimum-variance (UMV) filtering technique has been developed [4,8]. The problem is transformed into finding a gain matrix such that the trace of the estimation error matrix is minimized. Certain algebraic constraints must be satisfied for the unbiased estimator to exist.

In this paper, we address the state estimation problem of systems when the unknown inputs can be treated as a WSS process with rational power spectral density (PSD), while no other information about the unknown inputs is known. We propose a new unknown input filtering approach based on the system realization techniques. Instead of constructing the observer gain matrix which needs to satisfy certain constraints, we apply the standard Kalman filtering using the following procedure: 1) recover the statistics of the unknown inputs from the measurements by solving an appropriate least squares problem, 2) find a spectral factorization of unknown input process by fitting an autoregressive (AR) model, 3) construct an innovations model of the unknown inputs via the eigensystem realization algorithm (ERA) [11] to

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the recovered input correlation data, and 4) apply the ASKF for state estimation. To reduce the computational cost of the ASKF, we apply the Balanced Proper Orthogonal Decomposition (BPOD) technique [16] to construct a reduced order model (ROM) for filtering.

The main advantage of the AR model based algorithm we propose is that the performance of the algorithm is better than the ASKF, OTSKF and UMV algorithms when the unknown inputs can be treated as WSS processes with rational PSDs. The AR model based algorithm we propose constructs one particular realization of the true unknown input model, and the performance of the AR model based algorithm is the same as OTSKF when the assumed unknown input model used in OTSKF is accurate, and is better than UMV algorithm in the sense that the error covariances are smaller. With the increase of the sensor noise, we have seen that the performance of AR model based algorithm gets much better than the UMV algorithm.

The paper is organized as follows. In Section 2, the problem is formulated, and general assumptions are made about the system and the unknown inputs. In Section 3, the AR model based unknown input realization approach is proposed. The unknown input statistics are recovered from the measurements, then a linear model is constructed using an AR model and the ERA is used to generate a balanced minimal realization of the unknown inputs. After an innovations model of the unknown inputs is constructed, the ASKF is applied for state estimation in Section 4. Also, a ROM constructed using the BPOD is introduced to reduce the computational cost of Kalman filter. Section 5 presents a numerical example consisting of a stochastically perturbed heat transfer problem that utilizes the proposed technique.

2 Problem Formulation

Consider a complex valued linear time-invariant (LTI) discrete time system:

$$x_k = Ax_{k-1} + Bu_{k-1}, y_k = Cx_k + v_k, \tag{1}$$

where $x_k \in \mathbb{C}^n$, $y_k \in \mathbb{C}^q$, $v_k \in \mathbb{C}^q$, $u_k \in \mathbb{C}^p$ are the state vector, the measurement vector, the measurement white noise with zero mean and known covariance Ω, and the unknown stochastic inputs respectively. The process u_k is used to model the presence of the external disturbances, process noise, and unmodeled terms. Here, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{q\times n}$ are known.

Denote $h_i = CA^{i-1}B, i = 1, 2, \cdots$ as the Markov parameters of system (1). We use x^* to denote the complex conjugate transpose of x, and x^T to denote the transpose of x. Denote $\bar{h}_i$ as the matrix h_i with complex conjugated entries, and $h_i^* = (\bar{h}_i)^T$. $\|A\| = (\sum_{i,j=1}^n |a_{i,j}|^2)^{1/2}$ denotes the Frobenius norm of matrix A, and $\|x\|_2 = (|x_1|^2+|x_2|^2+\cdots+|x_n|^2)^{1/2}$ denotes the Euclidean norm of vector x.

The following assumptions are made about system (1):

- A1. A is a stable matrix, and (A, C) is detectable.
- A2. rank$(B) = p$, rank$(C) = q, p \leq q$ and rank (CAB) = rank $(B) = p$.
- A3. u_k and v_k are uncorrelated.
- A4. We further assume that the unknown input u_k can be treated as a WSS process:

$$\xi_k = A_e\xi_{k-1} + B_e\nu_{k-1}, u_k = C_e\xi_k + \mu_k, \tag{2}$$

where ν_k, μ_k are uncorrelated white noise processes.

Remark 1 *A2 is a weaker assumption than the so-called "observer matching" condition used in unknown input observer design. The observer matching condition requires rank (CB) = rank $(B) = p$, which in practice, may be too restrictive. A2 implies that if there are p inputs, then there should be at least p controllable and observable modes. A4 implies that u_k is a WSS process with a rational power spectrum.*

In this paper, we consider the state estimation problem when the system (2), i.e., (A_e, B_e, C_e) are unknown. Given the output data y_k, we want to construct an innovations model for the unknown stochastic input u_k, such that the output statistics of the innovations model and system (2) are the same. Given such a realization of the unknown input, we apply the standard Kalman filter for the state estimation, augmented with the unknown input states.

3 AR Model Based Unknown Input Realization Technique

In this section, we propose an AR model based unknown input realization technique which can construct an innovations model of the unknown inputs such that the ASKF can be applied for state estimation. First, a least squares problem is formulated based on the relationship between the inputs and outputs to recover the statistics of the unknown inputs. Then an AR model is constructed using the recovered input statistics, and a balanced realization model is then constructed using the ERA.

3.1 Extraction of Input Autocorrelations via a Least Squares Problem

Consider system (1) with zero initial conditions, the output y_k can be written as:

$$y_k = \sum_{i=1}^{\infty} h_i u_{k-i} + v_k. \tag{3}$$

For a LTI system, under assumption A1 that A is stable, the output $\{y_k\}$ is a WSS process when $\{u_k\}$ is WSS. The output autocorrelation can be written as:

$$R_{yy}(m) = E[y_k y^*_{k+m}] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_i u_{k-i} u^*_{k+m-j} h^*_j$$
$$+R_{vv}(m) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_i R_{uu}(m+i-j) h^*_j + R_{vv}(m), \quad (4)$$

where $m = 0, \pm 1, \pm 2, \cdots$ is the time-lag between y_k and y_{k+m}. Here, assumption A3 is used.

We denote $\hat{R}_{yy}(m) = R_{yy}(m) - R_{vv}(m)$, where $R_{vv}(m) = \Omega$ for $m = 0$, and $R_{vv}(m) = 0$, otherwise. Therefore, the relationship between input and output autocorrelation function is given by:

$$\hat{R}_{yy}(m) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_i R_{uu}(m+i-j) h^*_j. \quad (5)$$

To solve for the unknown input autocorrelations $R_{uu}(m)$, first we need to use a theorem from linear matrix equations [15].

Theorem 2 *Consider the matrix equation*

$$AXB = C, \quad (6)$$

where A, B, C, X are all matrices. If $A \in \mathbb{C}^{m\times n} = (a_1, a_2, \cdots, a_n)$, where a_i are the columns of A, then define $vec(A) \in \mathbb{C}^{mn\times 1}$ and the Kronecker product $A \otimes B$ as:

$$vec(A) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \dots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}. \quad (7)$$

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is an $mp \times nq$ block matrix.

The matrix equation (6) can be transformed into one vector equation:

$$(B^T \otimes A) vec(X) = vec(C), \quad (8)$$

where $B^T \otimes A$ is the Kronecker product of B^T and A.

Thus, by applying Theorem 2, (5) can be written as:

$$\underbrace{\text{vec}(\hat{R}_{yy}(m))}_{\in R^{q^2\times 1}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \underbrace{\bar{h}_j \otimes h_i}_{\in R^{q^2\times p^2}} \underbrace{\text{vec}(R_{uu}(m+i-j))}_{\in R^{p^2\times 1}}, \quad (9)$$

Now, we estimate the unknown input autocorrelations by the following procedure.

3.1.1 *Choose design parameter M*

Under assumption A1, i.e., the system is stable, the Markov parameters of the system (1) have the following property: $\|h_i\| \to 0$ as $i \to \infty$.

We choose a design parameter M, such that (9) can be written as:

$$\text{vec}(\hat{R}_{yy}(m)) = \sum_{i=1}^{M}\sum_{j=1}^{M} \bar{h}_j \otimes h_i \text{vec}(R_{uu}(m+i-j)). \quad (10)$$

where M varies with different systems and can be chosen as large as desired.

3.1.2 *Choose design parameters N_o, N_i*

Under assumption A1 and A4, $\|R_{uu}(m)\| \to 0$, and $\|\hat{R}_{yy}(m)\| \to 0$ as $m \to \infty$. As a standard method when computing a power spectrum from an autocorrelation function, we choose design parameters N_i and N_o, such that the input autocorrelations are calculated when $|m| \leq N_i$, and the output autocorrelations are calculated when $|m| \leq N_o$. The numbers N_o and N_i depend on the dynamic system and unknown inputs, and can be chosen as large as required. We have the following proposition.

Proposition 3 *The relation $N_i \leq N_o$ holds, which implies that all significant input autocorrelations can be recovered from the output autocorrelations.*

PROOF. The support of $\hat{R}_{yy}$ is limited to $(-N_o, N_o)$, thus, we have: $\hat{R}_{yy}(N_o + 1) = 0$. From (9),

$$\text{vec}(\hat{R}_{yy}(N_o+1)) = \sum_{i=1}^{\infty} \bar{h}_i \otimes h_i \text{vec}(R_{uu}(N_o+1))$$
$$+ \sum_{i=2}^{\infty} \bar{h}_{i-1} \otimes h_i \text{vec}(R_{uu}(N_o)) + \cdots . \quad (11)$$

If $N_i > N_o$, which means $R_{uu}(N_o+1) \neq 0$, then it follows that $R_{yy}(N_o+1)$ is also not negligible, which contradicts the assumption, and hence, as a consequence, $N_i \leq N_o$.

Thus, the following equation is used for computation of the unknown input autocorrelations. For $|m| \leq N_o$,

$$\text{vec}(\hat{R}_{yy}(m)) = \sum_{i=1}^{M}\sum_{j=1}^{M} \bar{h}_j \otimes h_i \text{vec}\underbrace{(R_{uu}(m+i-j))}_{|m+i-j|\leq N_i}, \quad (12)$$

3.1.3 Solve the least squares problem

We collect $2N_o+1$ output autocorrelations, and from the above assumptions, there are $2N_i+1$ unknown input autocorrelations:

$$\underbrace{\begin{pmatrix} \mathrm{vec}(\hat{R}_{yy}(-N_o)) \\ \mathrm{vec}(\hat{R}_{yy}(-N_o+1)) \\ \vdots \\ \mathrm{vec}(\hat{R}_{yy}(0)) \\ \mathrm{vec}\hat{R}_{yy}(1)) \\ \vdots \\ \mathrm{vec}(\hat{R}_{yy}(N_o)) \end{pmatrix}}_{\mathrm{vec}(\hat{R}_{yy})} = C_{yu} \underbrace{\begin{pmatrix} \mathrm{vec}(R_{uu}(-N_i)) \\ \vdots \\ \mathrm{vec}(R_{uu}(0)) \\ \mathrm{vec}(R_{uu}(1)) \\ \vdots \\ \mathrm{vec}(R_{uu}(N_i)) \end{pmatrix}}_{\mathrm{vec}(R_{uu})}, \quad (13)$$

where C_{yu} is the coefficient matrix and can be calculated from (12). Under assumption A1, A2 and A4, we have the following proposition.

Proposition 4 *Equation (13) has a unique least-squares solution $\hat{R}_{uu}(m), m = 0, \pm1, \pm2, \cdots, \pm N_i$.*

PROOF. We partition the matrix C_{yu} into three parts as $C_{yu} = \begin{pmatrix} C_t \\ C_m \\ C_b \end{pmatrix}$, where C_m contains the $q^2(N_o-N_i)+1, \cdots, q^2(N_o+N_i+1)$ rows of C_{yu} and can be expressed as:

$$C_m = \begin{pmatrix} \sum_{j=1}^{M} \bar{h}_j \otimes h_j & \sum_{j=1}^{M-1} \bar{h}_j \otimes h_{j+1} & \cdots & \cdots \\ \sum_{j=1}^{M-1} \bar{h}_{j+1} \otimes h_j & \sum_{j=1}^{M} \bar{h}_j \otimes h_j & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \sum_{j=1}^{M} \bar{h}_j \otimes h_j \end{pmatrix}. \quad (14)$$

In the following, we prove that $C_m \in \mathbb{C}^{q^2(2N_i+1)\times p^2(2N_i+1)}$ has full column rank $p^2(2N_i+1)$ by induction.

Let $N_i = 0$, then

$$C_m(0) = \sum_{j=1}^{M} \bar{h}_j \otimes h_j = (CV_{co} \otimes CV_{co})(I + \Lambda_{co} \otimes \Lambda_{co} + \cdots + \Lambda_{co}^{M-1} \otimes \Lambda_{co}^{M-1})(U'_{co}B \otimes U'_{co}B), \quad (15)$$

where Λ_{co} are the controllable and observable eigenvalues of A, and (V_{co}, U_{co}) are the corresponding right and left eigenvectors. Under the assumption A2, if rank $(CAB) = p$, and since $CAB = CV_{co}\Lambda_{co}U'_{co}B$, which implies that rank $(C_m(0)) = p^2$.

If rank $C_m(N_i-1)$ has rank $p^2(2N_i-1)$, then consider $C_m(N_i)$:

$$C_m(N_i) = \begin{pmatrix} C_m(0) & C_{12} & C_{13} \\ C_{21} & C_m(N_i-1) & C_{23} \\ C_{31} & C_{32} & C_m(0) \end{pmatrix}, \quad (16)$$

where $C_{12}, C_{13}, C_{21}, C_{23}, C_{31}, C_{32}$ are some matrices, and it can be proved that $C_m(N_i)$ has $p^2 + p^2(2N_i-1) + p^2 = p^2(2N_i+1)$ independent columns, and hence, rank $(C_m(N_i)) = p^2(2N_i+1)$.

Thus, by induction, C_m has full column rank, and hence, C_{yu} has full column rank. Since $q \geq p$, it is an overdetermined system, so there exists a unique solution to the least squares problem.

Remark 5 *The size of C_{yu} is $q^2(2N_o+1) \times p^2(2N_i+1)$ and it would be large when p and q increase, and hence, large scale least squares problem needs to be solved for systems with large number of inputs/outputs. For example, a modified conjugate gradients method [2] could be used as follows.*

The least squares problem need to be solved is: $vec(\hat{R}_{yy}) = C_{yu} vec(R_{uu})$, and multiply C_{yu}^* on both sides: $C_{yu}^* vec(\hat{R}_{yy}) = C_{yu}^* C_{yu} vec(R_{uu})$. If we denote $L_s = C_{yu}^* vec(\hat{R}_{yy})$, $\bar{x} = vec(R_{uu})$, and $C_s = C_{yu}^* C_{yu}$, then $C_s = C_s^*$, and the problem is equivalent to solve the least squares problem for $\bar{x}$: $C_s\bar{x} = L_s$, and a conjugate gradient method to solve this problem is summarized in Algorithm 1.

Algorithm 1 Conjugate gradient algorithm

(1) For a least-squares problem $C_s\bar{x} = L_s$, where $C_s = C_s^*$, $\bar{x}$ is unknown.
(2) Start with a random initial solution $\bar{x}_0$.
(3) $r_0 = L_s - C_s\bar{x}_0$, $p_0 = r_0$.
(4) for $k = 0$, repeat
(5) $\alpha_k = \frac{r_k^* r_k}{p_k^* C_s p_k}$,
$\bar{x}_{k+1} = \bar{x}_k + \alpha_k p_k$,
$r_{k+1} = r_k - \alpha_k C_s p_k$,
if r_{k+1} is sufficient small then exit loop.
$\beta_k = \frac{r_{k+1}^* r_{k+1}}{r_k^* r_k}$,
$p_{k+1} = r_{k+1} + \beta_k p_k$,
$k = k + 1$,
end repeat.
(6) The optimal estimation is x_{k+1}.

The error of the input autocorrelations we extract results from two design parameters: the choice of M and N_o, N_i. The following proposition considers the total errors of input autocorrelations we recover.

Proposition 6 *Denote $R_{uu}(m)$ as the "true" input autocorrelations, $\hat{R}_{uu}(m)$ as the input autocorrelation function we estimate from the output autocorrelations, and let $\Delta(m) = R_{uu}(m) - \hat{R}_{uu}(m)$ be the error between the estimated input autocorrelation and the "true" input autocorrelation. We assume that $\|h_i\| \leq \delta, i > M$, $\|R_{uu}(m)\| \leq \delta, |m| > N_i$, and $\|\hat{R}_{yy}(m)\| \leq \delta, |m| > N_o$ where δ is small enough. Then $\|\Delta(m)\| \leq k\delta$, where k is some constant.*

The perturbation theory [13] is used to prove the above result, and the proof is shown in the technical report [19]. The results above show that if M, N_i, N_o are chosen large enough, the errors in estimating the input autocorrelations can be made arbitrarily small.

3.2 Construction of the AR Based Innovations Model

After we extract the input autocorrelations from the output autocorrelations, we want to construct a system which will generate the same statistics as the ones we recovered in Section 3.1. If assumption A4 is satisfied, i.e., $\{u_k\}$ is WSS with a rational power spectrum, the power spectrum of u_k is continuous, and can be modelled as the output of a casual linear time invariant system driven by white noise [18]. Such system can be constructed by using an autoregressive moving average (ARMA) model, and in practice, a MA model can often be approximated by a high-order AR model, and thus, with enough coefficients, any stationary process can be well approximated by using either AR or MA models (Chapter 9, [17]), and in this paper, we use an AR model to fit the data. In an AR model, the time series can be expressed as a linear function of its past values, i.e.,

$$u(k) = \sum_{i=1}^{M_i} a_i u(k-i) + \epsilon(k), \tag{17}$$

where $\epsilon(k)$ is white noise with distribution $N(0, \Omega_r)$, M_i is the order of the AR model, and $a_i, i = 1, 2, \cdots, M_i$ are the coefficient matrices. For a vector autoregressive model with complex values, the Yule-Walker equation [3] which is used to solve for the coefficients needs to be modified. The modified Yule-Walker equation can be written as:

$$\begin{pmatrix} R_{uu}(-1) & R_{uu}(-2) & \cdots & R_{uu}(-M_i) \end{pmatrix} = \begin{pmatrix} a_1^* \\ a_2^* \\ \cdots \\ a_{M_i}^* \end{pmatrix}^* \times \begin{pmatrix} R_{uu}(0) & \cdots & R_{uu}(1-M_i) \\ R_{uu}(1) & \cdots & R_{uu}(2-M_i) \\ \vdots & \vdots & \vdots \\ R_{uu}(M_i - 1) & \cdots & R_{uu}(0) \end{pmatrix}. \tag{18}$$

Equation (18) is used to solve for the coefficient matrices $a_i, i = 1, 2, \cdots, M_i$. The covariance of the residual white noise $\epsilon(k)$ can be solved using the following equation:

$$R_{\epsilon\epsilon}(m) = R_{uu}(m) - \sum_{i=1}^{M_i}\sum_{j=1}^{M_i} a_i R_{uu}(m+i-j) a_j^*, \tag{19}$$

where $\Omega_r = R_{\epsilon\epsilon}(0)$. The balanced minimal realization for the AR model (17) can be expressed as:

$$\eta_k = A_n \eta_{k-1} + B_n u_{k-1}, u_k = C_n \eta_k + \epsilon_k, \tag{20}$$

where (A_n, B_n, C_n) are solved by using the ERA technique [11] with $a_i, i = 1, \cdots, M_i$ as the Markov parameters of the system.

Equation (20) is equivalent to:

$$\eta_k = (A_n + B_n C_n)\eta_{k-1} + B_n \epsilon_{k-1}, u_k = C_n \eta_k + \epsilon_k, \tag{21}$$

where ϵ_k is white noise with covariance Ω_r. We make the following remark.

Remark 7 *We need to find a stable $A_n + B_n C_n$ in (21). In practice, we calculate the Markov parameters of system (21) using $a_i, i = 1, \cdots, M_i$ first, and then use the ERA for the state space realization. If the Markov parameters of system (21) are $\hat{a}_i, i = 1, \cdots, M_i$, then $\hat{a}_1 = C_n B_n = a_1, \hat{a}_2 = C_n(A_n + B_n C_n)B_n = a_2 + a_1 a_1, \cdots$. As we explained before, for a WSS process with rational power spectrum, from [18] , we can always find a stable realization $(A_n + B_n C_n, B_n, C_n)$.*

By using the Cholesky decomposition, we can find a unique lower triangular matrix P such that: $\Omega_r = PP^*$. If w_k is white noise with distribution $N(0, 1)$, then Pw_k would be white noise with distribution $N(0, \Omega_r)$. Thus, the innovations model we construct that has the same statistics as the unknown input system (2) is:

$$\begin{aligned} \eta_k = (A_n + B_n C_n)\eta_{k-1} + B_n P w_{k-1}, \\ u_k = C_n \eta_k + P w_k, \end{aligned} \tag{22}$$

where w_k is a randomly white noise with standard normal distribution.

Under assumption A4, we have the following proposition.

Proposition 8 *Denote $\hat{R}_{uu}(m)$ as the input autocorrelations recovered from the measurements, then $\hat{R}_{uu}(m)$ can be reconstructed exactly by using the innovations model (22), i.e., $\tilde{R}_{uu}(m) = \hat{R}_{uu}(m)$, where $\tilde{R}_{uu}(m)$ is the input autocorrelations of the realization of system (22).*

From Proposition 6 and 8, under the same assumptions, the following corollary immediately follows.

Corollary 9 *Denote u_k as the actual unknown input process, and $R_{uu}(m)$ as the actual input autocorrelation function. Then $\|\tilde{R}_{uu}(m) - R_{uu}(m)\| \le k_a\delta$, where k_a is some constant, when δ is small enough. System (22) is an innovations model for the unknown input u_k.*

The procedure of constructing the innovations model is summarized in Algorithm 2.

Remark 10 *A generalization to the joint state and unknown input estimation.*

When the unknown inputs affect both the states and outputs, i.e.

$$x_{k+1} = Ax_k + Bu_k, y_k = Cx_k + Du_k + v_k, \tag{23}$$

where u_k is the stochastic unknown input, v_k is the measurement noise. The solution y_k can be written as:

$$y_k = \sum_{i=1}^{M} h_i u_{k-i} + Du_k + v_k, \tag{24}$$

and the relationship between output autocorrelations and input autocorrelations is:

$$vec(\hat{R}_{yy}(m)) = \sum_{i=1}^{M}\sum_{j=1}^{M} \bar{h}_j \otimes h_i vec(R_{uu}(m+i-j)) + \sum_{i=1}^{M} \bar{D} \otimes h_i vec(R_{uu}(m+i)) + \sum_{i=1}^{M} \bar{h}_j \otimes D vec(R_{uu}(m-j)) + \bar{D} \otimes D vec(R_{uu}(m)) \tag{25}$$

where $\hat{R}_{yy}(m) = R_{yy}(m) - R_{vv}(m)$. Notice that the first term is the same as (12), and the last three terms correspond to the perturbation of the unknown inputs in the outputs. It can also be formulated as a least squares problem (13), and an unknown input system may be realized following the same procedure as in Algorithm 2.

Remark 11 *For real valued system, we can save the computation by using the properties of autocorrelation functions:*

$$R_{u_iu_i}(-m) = R_{u_iu_i}(m),$$
$$R_{u_iu_j}(-m) = R_{u_ju_i}(m), i \neq j \tag{26}$$

Thus, we only need to collect $N_o + 1$ output autocorrelations and have $p^2(N_o + 1)$ equations with $q^2(N_i + 1)$ unknowns in (13).

Algorithm 2 AR model based unknown input realization technique

(1) Choose a finite number N_o, compute output autocorrelation function $R_{yy}(m)$ by using measurements y_k, $|m| \le N_o$.
(2) Choose a finite number M, construct the coefficient matrix C_{yu} from (12).
(3) Choose a finite number N_i, solve the least squares problem (13) for unknown input autocorrelation function $R_{uu}(m)$, $|m| \le N_i$.
(4) Construct an AR model for the unknown input $u(k) = \sum_{i=1}^{M_i} a_i u(k-i) + \epsilon(k)$, find the coefficient matrices $a_i, i = 1, 2, \cdots M_i$ by solving the modified Yule-Walker equation (18).
(5) Find the covariance Ω_r of $\epsilon(k)$ by solving (19).
(6) Construct the state space representation (20) for the AR model using ERA.
(7) Find a unique lower triangular matrix P such that $\Omega_r = PP^*$, and construct an innovations model as in (22).

4 Augmented State Kalman Filter and Model Reduction

After we construct an innovations model for the unknown inputs, we apply the standard Kalman filter on the augmented system with states augmented by the unknown input states. A ROM based filter is also constructed using the BPOD for reducing the computational cost of the resulting filter.

4.1 Augmented State Kalman Filter

The full order system can be represented by augmenting the states of the original system as:

$$\begin{pmatrix} x_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} A & BC_n \\ 0 & A_n + B_nC_n \end{pmatrix} \begin{pmatrix} x_k \\ \eta_k \end{pmatrix} + \begin{pmatrix} BP \\ B_nP \end{pmatrix} w_k,$$
$$y_k = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x_k \\ \eta_k \end{pmatrix} + v_k, \tag{27}$$

where w_k is white noise with standard normal distribution. v_k is white noise with known covariance.

Remark 12 *The augmented state system (27) is stable and detectable. The eigenvalues of the augmented system (27) are the eigenvalues of A and the eigenvalues of $A_n + B_nC_n$. From assumption A1, A is stable, from Remark 7, $A_n + B_nC_n$ is stable, and hence, the augmented system (27) is stable. From assumption A1, system (1) is detectable, and from the asymptotic stability of matrix $A_n + B_nC_n$, (21) is also detectable, therefore, all the unobservable modes in (27) are asymptotically stable, which implies that (27) is detectable. Thus, we may now use the standard Kalman filter for state estimation of the augmented system (27).*

4.2 Unknown Input Estimation Using Model Reduction

For large scale systems as we would like to consider, in particular, the systems governed by partial differential equations (suitably discretized), we can use model reduction technique such as Balanced Proper Orthogonal Decomposition (BPOD) [16] to construct a ROM first, and then extract the input autocorrelations from the reduced order model. We apply the Kalman filter to the ROM to reduce the computational cost.

The ROM system is extracted from the full order system using the BPOD and is denoted by:

$$x_k = A_r x_{k-1} + B_r u_{k-1},$$
$$y_k = C_r x_k + v_k. \tag{28}$$

The BPOD algorithm can be found in [16].

Let $\hat{h}_i = C_r A_r^{i-1} B_r, i = 1, 2, \cdots, M$ be the Markov parameters of the ROM. Then the relationship between input autocorrelations and output autocorrelations can be written as:

$$\hat{R}_{yy}(m) = \sum_{i=1}^{M} \sum_{j=1}^{M} \hat{h}_i R_{uu}(m + i - j) \hat{h}_j^*. \tag{29}$$

Following the same procedure as in Algorithm 2, we can now recover the input autocorrelations, and construct an innovations model which can generate the same statistics as the unknown inputs. The advantage of using model reduction is that for a large scale system, computing $\hat{h}_i = C_r A_r^{i-1} B_r$ is much faster than computing $h_i = CA^{i-1}B$ because of the reduction in the size of A. Also, the order of the ROM is much smaller than the order of the full order system, and thus the computational cost of using the Kalman filter is much reduced. Hence, even with the augmented states, the standard Kalman filter remains computationally tractable.

Remark 13 *To reduce the computational cost of the augmented states in Kalman filter, we can also use the existing optimal two-stage or three-stage Kalman filtering technique [5, 10], which decouple the augmented filter into two parallel reduced order filters. These techniques are preferable when the order of the innovations model is high, while the BPOD based ROM filter is preferable when the order of the dynamic system is high.*

5 Computational Results

We test the method on the perturbed heat equation. We construct the unknown input system by using both the full order system as well as the ROM constructed by BPOD. We check the results by comparing the autocorrelation functions of the inputs, outputs and the states. Also, we show the state estimation using the Kalman filter.

The equation for heat transfer by conduction along a slab is given by the partial differential equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f,$$
$$T|_{x=0} = 0, \frac{\partial T}{\partial x}|_{x=L} = 0, \tag{30}$$

where α is the thermal diffusivity, $L = 1m$, and f is the unknown forcing. There are two point sources located at $x = 0.5m$ and $x = 0.6m$.

The system is discretized using finite difference approach, and there are 50 grids which are equally spaced. To satisfy the observer matching condition in the UMV algorithm, we take two measurements at $x = 0.5m$, $x = 0.6m$. The measurement noise is white noise with covariance $0.1I_{2\times2}$. In the simulation, the unknown inputs are generated using (2) with

$$A_e = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.2 \end{pmatrix}, B_e = C_e = I_{2\times2}, \tag{31}$$

and $\nu_k = 0, \mu_k \sim N(0, 10I_{2\times2})$. The design parameters $M = 4000$, $N_i = 200$, $N_o = 2000$ are chosen as follows. M is chosen so that the Markov parameters $\|h_i\| \approx 0, i > M$. N_i and N_o are chosen by trial and error. First, we randomly choose a suitable N_i and N_o, where $N_i \leq N_o$. Then we follow the AR based unknown input realization procedure, and construct the augmented state system (27). Given the white noise processes w_k, v_k perturbing the system, we check the output statistics of the augmented state system (27). If the errors are small enough, we stop, otherwise, we increase the values of N_i and N_o, and repeat the same procedure until the errors are negligible. Notice that increasing M, N_i, N_o would increase the accuracy of the input statistics we can recover, but also increases the computational cost.

First, in Figure 1, we show the comparison of the input correlations we recover with the actual input correlations. Since there are two inputs, thus, the cross-correlation function between input 1 and input 2 are also included. It can be seen that the statistics of the unknown inputs can be recovered almost perfectly, and given the system perturbed by the unknown inputs innovations model we constructed, the statistics of the outputs and the states are almost the same as well.

Next, we compare the performance of the unknown inputs constructed using the ROM with the full order system. The full order system has 50 states, and the ROM has 20 states. The relative error of the input correlation is shown in Figure 2. We can see that the statistics re-

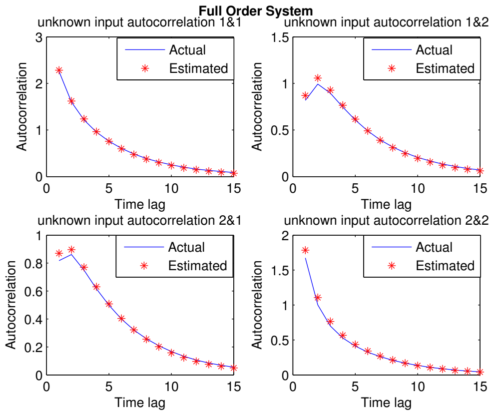

Fig. 1. Comparison of input autocorrelations

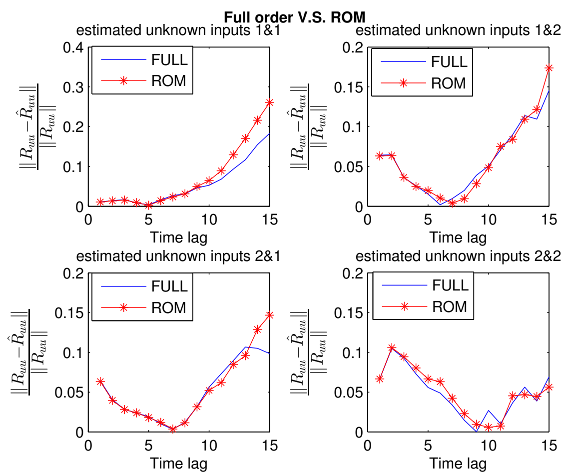

Fig. 2. Comparison of input autocorrelation relative error

constructed by using the ROM is not as accurate as using the full order system, however, the relative error is on the same scale, and hence, the computational cost is reduced without losing much accuracy.

The state estimation using ROM is shown in Figure 3. We randomly choose two states and show the comparison of the actual state with the estimated states. The state estimation error and 3σ bounds are shown. It can be seen that the Kalman filter using the ROM performs well, and hence, for a large scale system, the computational complexity of ASKF can be reduced by using the BPOD.

5.1 Comparison with OTSKF and UMV Algorithms

Next, we compare the performances of the AR model based algorithm with OTSKF and UMV algorithms. The OTSKF and UMV algorithms we use can be found in [7].

The assumed unknown input model used in the OTSKF is not the same as the true model, in particular, the

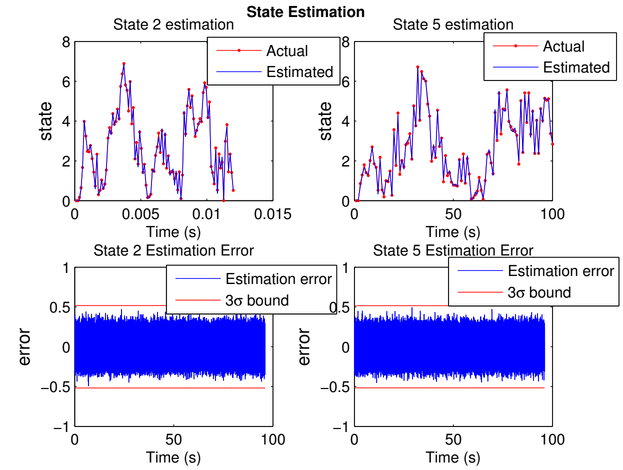

Fig. 3. Comparison of state estimation

system matrices of the input system are perturbed from the true values, the model used for OTSKF is:

$$\eta_{k+1} = A_o \eta_k + v_k = \begin{pmatrix} 0.4569 & 0.2768 \\ 0.2214 & 0.4016 \end{pmatrix} \eta_k + v_k, \qquad (32)$$

where $v_k \sim N(0, 10I_{2\times 2})$. Here, A_o is chosen as follows. The eigenvalues of A_e in (31) are $0.7, -0.2$. We perturb the eigenvalues of A_e with randomly generated numbers between $[-0.3, 0.3]$ and $[-0.8, 0.8]$ with uniform distribution respectively, and keep the eigenvectors same as the eigenvectors of A_e. The perturbed eigenvalues are $0.6783, 0.1802$. We calculate the output statistics of (31) and (32), and we can see that the unknown input statistics used in OTSKF are perturbed by 5% about the true value. The estimation of the initial state $\bar{x}_0$ and covariance $\bar{P}_0$ in three algorithms are the same.

Denote the average root mean square error(ARMSE) as:

$$ARMSE = \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{\sum_{k=1}^{n} (\hat{x}_i(k) - x_i(k))^2}{n}}, \qquad (33)$$

where $\hat{x}_i(k)$ is the state estimate $\hat{x}_i$ at time t_k, and $x_i(k)$ is the true state x_i at time t_k, where i denotes the i^{th} component of the state vector.

Suppose at the state component x_i, the measurement noise v_k is a white noise with zero mean and covariance Ω_i. We define a noise to signal ratio (NSR):

$$NSR = \sqrt{\frac{|\Omega_i|}{(E[x_i x_i^*])}}. \qquad (34)$$

We vary the measurement noise covariance Ω_i, and for each Ω_i, a Monte Carlo simulation of 10 runs is performed to compare the magnitude of the ARMSE us-

Table 1
Performances of the AR model based algorithm, OTSKF and UMV

NSR	AR model based	OTSKF	UMV
0.2215%	0.0036	0.0111	0.0033
6.8704%	0.0832	0.2418	0.0874
13.5171%	0.1309	0.3955	0.1528
20.3456%	0.3810	0.6516	0.4332
26.9467%	0.4190	0.7141	0.5112

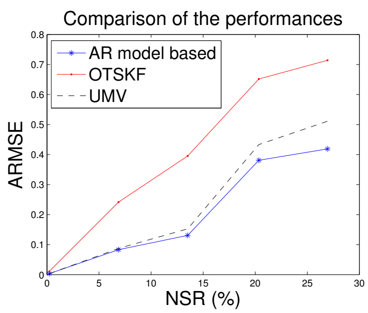

Fig. 4. Comparison of the performances

ing AR model based algorithm with the OTSKF and UMV algorithms in Table 1. The comparison is shown in Figure 4. It can be seen that the AR model based method performs the best. Note that when the assumed unknown input model used in OTSKF is not accurate, the performance of AR model based algorithm is much better while with increase in the sensor noise, the performance of the AR model based algorithm gets better than the UMV algorithm. It should also be noted that when the sensors and the unknown inputs are non-collocated, the "observer matching" condition is not satisfied, and hence, the UMV algorithm can not be used, while the OTSKF and the AR model based algorithm are not affected.

6 Conclusion

In this paper, we have proposed a balanced unknown input realization method for the state estimation of system with unknown inputs that can be treated as a wide sense stationary process. We recover the unknown inputs statistics from the output data using a least squares procedure, and then construct a balanced minimal realization of the unknown inputs using an AR model and the ERA technique. The recovered innovations model is used for the state estimation, and the standard Kalman filter is applied on the augmented system. We have compared the performances of the AR model based algorithm with the OTSKF and UMV algorithms. In practice, we have also seen that the proposed algorithm can be used to estimate the locations of the unknown inputs, so the next step of our work will require us to generalize the algorithm to estimate the locations of the unknown inputs in a rigorous fashion.

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