# On the Probabilistic Completeness of the Sampling-based Motion Planning Methods Under Uncertainty

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*Abstract*— **This paper extends the concept of probabilistic completeness defined for the motion planners in the absence of noise, to the concept of "probabilistic completeness under uncertainty" for the motion planners that perform planning in the presence of uncertainty. According to the proposed definition, an approach is proposed to verify the probabilistic completeness under uncertainty. Finally, it is shown that the sampling-based method FIRM [1] is a probabilistically complete algorithm under uncertainty.**

## I. Introduction

Motion planning for a moving object in the presence of obstacles is one of the main challenges in robotics, and has attracted immense attention over the last two decades [2], [3], [4]. Sampling-based methods are one of the successful methods in solving many planning problems. Sampling-based methods were initially developed for the motion planning in the absence of noise, (e.g., [5], [6], and [7]), and later were generalized to motion planning methods in the presence of the uncertainty (e.g., [1], [8], [9], [10]).

Due to the success of sampling-based methods in many practical planning problems, many researchers have investigated the theoretical basis for this success. However, almost all of these investigations have been done for algorithms that are designed for planning in the absence of uncertainty. The literature in this direction falls into two categories: path isolation-based methods and space covering-based methods.

*Path isolation-based analysis:* In this approach, one path is chosen, and it is tiled with some sets such as $\epsilon$-balls in [11] or sets with arbitrary shapes but strictly positive measure in [12]. Then the success probability is analyzed by investigating the probability of sampling in each of the sets that tile the given path, in the obstacle-free space. Methods in [11], [12], [13], and [14] are among the methods that perform path isolation-based analysis of planning algorithm.

*Space Covering-based analysis:* In the space covering-based analysis approach, the adequate number of sampled points to find a successful path is expressed in terms of a parameter $\epsilon$, which is a property of the environment. A space is $\epsilon$-good, if every point in the state space can at least be connected to an $\epsilon$ fraction of the space, using local planners. Methods [15] and [16] are among these methods.

These methods are developed for the case that the solution of planning algorithm is a path. However, in the presence of uncertainty, the concept of "successful path" is no longer meaningful, because on a given path, different policies may result in different success probabilities, some interpreted as successful, some not. Thus, since the planning algorithm returns a policy instead of a path, the success has to be defined for a policy. This paper extends these concepts to probabilistic spaces, i.e., to sampling-based methods concerning planning under uncertainty. We define the concept of successful policy and the concept of globally successful policy and formulate them rigorously.

Accordingly, we generalize the conventional concept of "probabilistic completeness" defined for motion planning methods in the absence of uncertainty to the concept of "probabilistic completeness under uncertainty", defined for the planners in the presence of uncertainty. According to this definition, we prove that *Feedback controller-based Information-state Roadmap Method* (FIRM), the planning algorithm proposed in [1], is a probabilistically complete algorithm. Also, the procedure used in this proof, provides some tools that can be used in analyzing planning methods under uncertainty.

In the next section, we first review the general formulation for the planning problem under uncertainty, and briefly explain the FIRM method [1]. In Section III we extend the concept of probabilistic completeness and define the concept of "probabilistic completeness under uncertainty". In Section IV, it is proved that FIRM [1] algorithm is probabilistically complete under uncertainty. We conclude the paper in section VI.

## II. Motion Planning Under Uncertainty

Mainly, uncertainty in planning originates from the lack of exact knowledge on robot's motion model, robot's sensing model, and environment model, which are referred to as motion uncertainty, sensing uncertainty, and map uncertainty, respectively. In this paper, we focus on the motion and sensing uncertainty, but some of the concepts are extendible to the problems with a map uncertainty. Markov Decision Process (MDP) problem and Partially Observable MDP (POMDP) are the most general formulations, respectively, for the planning problem under motion uncertainty and for the planning problem under both motion and sensing uncertainty.

While in the deterministic setting, we seek an optimal path as the solution of motion planning problem, in the stochastic setting we seek an optimal feedback (mapping) $\pi$ as the solution of motion planning problem. $\pi$ as the solution of MDP, is a mapping from the state space to the control space and $\pi$ as the solution of POMDP is a mapping from the belief space to the control space. In the rest of paper, we

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focus on the POMDPs, as it is more general. However, all statements can be easily stated for the MDPs.

### A. POMDP

In solving POMDP problems, we deal with following components:

- $\mathbb{X}$ is the state space of the problem, containing all possible states of the system, $X \in \mathbb{X}$.
- $\mathbb{U}$ is the control space of the problem, containing all possible control inputs, $u \in \mathbb{U}$.
- $\mathbb{Z}$ is the observation space of the problem, containing all possible observation, $z \in \mathbb{Z}$.
- $b_k$ is the belief at the $k$-th step, which is the pdf of system state condition on the obtained measurements up to $k$-th time step, $b_k = p(X_k|z_{0:k})$.
- $\mathbb{B}$ is the belief space of the problem, containing all possible beliefs, $b \in \mathbb{B}$.
- $p(X'|X,u)$ and $p(b'|b,u)$ are the state and belief transition pdf's, respectively. Also, $p(z|X)$ is the observation pdf condition on the system's state.
- $c(b,u)$ is the cost of taking control $u$ at belief $b$.
- $\pi(\cdot) : \mathbb{B} \rightarrow \mathbb{U}$ is the solution of POMDP, which is a mapping (feedback) that assigns a control action for every belief in belief space. It is well known that the infinite horizon POMDP problem can be cast as a belief MDP problem [4], [17], whose solution is obtained by solving the following stationary Dynamic Programming (DP) equation for all $b$ on the belief space $\mathbb{B}$ [4], [17]:

$$J(b) = \min_u \{c(b,u) + \int_{\mathbb{B}} p(b'|b,u)J(b')db'\}, \quad \text{(1a)}$$

$$\pi(b) = \arg\min_u \{c(b,u) + \int_{\mathbb{B}} p(b'|b,u)J(b')db'\}, \quad \text{(1b)}$$

- $J(\cdot) : \mathbb{B} \rightarrow \mathbb{R}$ is called the cost-to-go (or value) function, that assigns a cost-to-go for every belief in belief space.
- $\Pi$ is the set of admissible policies. The mapping $\pi$ lives in the function space and can have extremely complex formats. Thus, often, in solving POMDPs the set of admissible policies, i.e., $\Pi$ is chosen as a rich subset of this huge space, on which the optimization in Eq.(1) is carried out.

### B. FIRM

In this subsection, we focus on the FIRM framework [1] for planning under uncertainty, and briefly explain it. Solving POMDPs over continuous state, control, observation, and belief spaces, and finding the best feedback $\pi \in \Pi$ is a challenge, in particular in the presence of state constraint, e.g. obstacles in the environments. Inspired by the sampling-based methods, FIRM samples local controllers, whose combination results in a feedback controller $\pi \in \Pi$ that approximates the solution of Eq.(1), over continuous spaces in the presence of obstacles.

The FIRM graph is a generalization of the PRM graph, whose nodes are small subsets of belief space and whose edges are Markov chains induced by feedback controllers. As a result, planning on FIRM is a Markov Decision Process (MDP) in the belief space, which is defined on FIRM nodes (a finite set), and thus it can be solved using standard Dynamic Programming (DP) techniques [17].

In the following, we briefly explain the FIRM framework and the elements used in its construction:

- $F$ and $\mathbb{X}_{free}$ denote the obstacles and obstacle-free parts of the state space, such that $F \cap \mathbb{X}_{free} = \emptyset$ and $F \cup \mathbb{X}_{free} = \mathbb{X}$.
- $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^{N_v}$ and $\mathcal{E} = \{\mathbf{e}_{ij}\}$ are the set of nodes and edges of the PRM that underlies FIRM. All PRM nodes and edges lie in $\mathbb{X}_{free}$.
- $\mu^j(\cdot;\mathbf{v}_j) : \mathbb{B} \rightarrow \mathbb{U}$ is the $j$-th local controller, parametrized by the $j$-th PRM node $\mathbf{v}_j$. Local controller is a mapping from belief space to the control space.
- $\mathbb{M} = \{\mu^j\}_{j=1}^{N_v}$ is the set of sampled local controllers, in FIRM. Note that the number of local controllers is the same as the number of PRM nodes.
- $\mathbb{M}(i) = \{\mu^j \in \mathbb{M} | \exists \mathbf{e}_{ij} \in \mathcal{E}\}$ denotes the set of local controllers that can be invoked from the node $B_i$. Apparently, $\mathbb{M}(i) \subset \mathbb{M}$.
- $\mathbb{X}_h = \mathbb{X} \times \mathbb{B}$ is referred to as hyper-state (or h-state) space that contain all possible h-states (state-belief pairs), $\mathcal{X} = (X,b) \in \mathbb{X}_h$.
- $p^\mu(X'|X)$, $p^\mu(b'|b)$, and $p^\mu(\mathcal{X}'|\mathcal{X})$ are the transition pdf's induced by the local controller $\mu$, over the state, belief, and h-state spaces, respectively. In this paper, $p^\mu(\cdot|\cdot)$ and $p(\cdot|\cdot,\mu)$ are used interchangeably.
- $\mathbb{P}_n(\cdot|b,\mu) : \sigma(\mathbb{B}) \rightarrow \mathbb{R}_{\geq 0}$ is the probability measure over the belief space, induced by the local controller $\mu$ after $n$ steps, starting from the belief $b$. Set $\sigma(\mathbb{B})$ is the Borel sigma-algebra of the belief space $\mathbb{B}$.
- $\mathbb{P}_n(\cdot|\mathcal{X},\mu) : \sigma(\mathbb{X}_h) \rightarrow \mathbb{R}_{\geq 0}$ is the probability measure over the h-state space, induced by the local controller $\mu$ after $n$ steps, starting from the h-state $\mathcal{X}$. Set $\sigma(\mathbb{X}_h)$ is the Borel sigma-algebra of the h-state space $\mathbb{X}_h$.
- $B_j$ is the $j$-th FIRM node, which is a subset of belief space, i.e., $B_j \subset \mathbb{B}$ and $B_j \in \sigma(\mathbb{B})$. The condition on $B_j$, in designing a FIRM, is that $B_j$ has to have non-zero probability measure under $\mu^j$ after some finite number of steps $N < \infty$, i.e. $\mathbb{P}_n(B_j|b,\mu^j) > 0$, for $n \geq N$ and for all $b$. Note that also $B_i \cap B_j = \emptyset$ for $i \neq j$.
- $\mathbb{V}$ is the set of all FIRM nodes, i.e., $\mathbb{V} = \{B_i\}_{i=1}^{N_v}$ and thus $B_i \in \mathbb{V}$.
- $\mathbb{P}(B_j|b,\mu^j) = \sum_{n=0}^{\infty} \mathbb{P}_n(B_j|b,\mu^j)$ is the probability of landing in $B_j$ before hitting obstacles $F$, and $\mathbb{P}(F|b,\mu^j)$ is the probability of colliding with obstacles $F$ before landing in $B_j$, both under the controller $\mu^j$ taken at $b$.
- $C^{\mu^j}(b)$ represents the expected cost of invoking local controller $\mu^j(\cdot)$ starting at belief state $b$ until the local controller stops executing. Mathematically:

$$C^{\mu^j}(b) = \sum_{t=0}^{\mathcal{T}} c(b_t, \mu^j(b_t)|b_0 = b), \quad (2)$$

$$\mathcal{T}^{\mu^j}(b) = \inf_t \{t | b_t \in B_j, b_0 = b\}. \quad (3)$$

where $\mathcal{T}^{\mu^j}$, which is a function of initial belief, is a random stopping time denoting the time at which the belief state enters the node $B_j$ under the controller $\mu^j$.

- $b^i_s$ is a particular point in $B_i$. For a sufficiently small $B_i$, any point in $B_i$ can be considered as $b^i_s$. If $B_i$ is a ball, usually the centre point is chosen as the $b^i_s$.
- $\pi^g(\cdot) : \mathbb{V} \to \mathbb{M}$ is a mapping over the FIRM graph, from FIRM nodes into the set of local controllers.

$$J^g(B_i) = \min_{\mathbb{M}(i)} C^{\mu^j}(B_i) + J(F)\mathbb{P}(F|B_i, \mu^j) + J^g(B_j)\mathbb{P}(B_j|B_i, \mu^j), \tag{4a}$$

$$\pi^g(B_i) = \arg\min_{\mathbb{M}(i)} C^{\mu^j}(B_i) + J(F)\mathbb{P}(F|B_i, \mu^j) + J^g(B_j)\mathbb{P}(B_j|B_i, \mu^j), \tag{4b}$$

where, $\mathbb{P}(B_j|B_i, \mu^j) := \mathbb{P}(B_j|b^i_s, \mu^j)$ and $\mathbb{P}(F|B_i, \mu^j) := \mathbb{P}(F|b^i_s, \mu^j)$. Also, $C^{\mu^j}(B_i) := C^{\mu^j}(b^i_s)$.

- $J^g(\cdot) : \mathbb{V} \to \mathbb{R}$ is the cost-to-go function over the FIRM nodes, that assigns a cost-to-go for every FIRM node $B_i$. $J(F)$ is some suitable user-defined cost for hitting obstacles.
- $\pi$ is the overall feedback generated using FIRM, by combining the policy $\pi^g$ on the graph and the local controllers $\mu^j$s. When a local controller $\mu$ is chosen using $\pi^g$, the local controller starts generating the controls based on the current belief at each time step, until the belief reaches the corresponding stopping region, denoted by $B(\mu)$. For example if the controller $\mu^j$ is chosen, the stopping region is $B_j$, i.e., $B_j = B(\mu^j)$. In the presence of obstacles, the collision with obstacles, i.e., $X \in F$, stops the execution of controllers in all levels.

$$\pi : \mathbb{B} \to \mathbb{U}, \tag{5}$$

$$u_k = \pi(b_k) = \begin{cases} \mu_k(b_k), \ \mu_k = \pi^g(b_k), & \text{if } b_k \in B(\mu_{k-1}) \\ \mu_k(b_k), \ \mu_k = \mu_{k-1}, & \text{if } b_k \notin B(\mu_{k-1}) \end{cases}$$

where, $\mu_k \in \mathbb{M}$ denotes the active local controller at time step $k$. The initial local controller is:

$$\mu_0(\cdot) = \begin{cases} \arg\min_{\mathbb{M}} C^{\mu^j}(b_0) + J(F)P^{\mu^j}(F|b_0) \\ \qquad + J(B_j)P^{\mu^j}(B_j|b_0), & \text{if } \ b \notin \bigcup_m B_m \\ \pi^g(b_0), & \text{if } \ b \in \bigcup_m B_m \end{cases} \tag{6}$$

- $\Pi$ is the set of admissible policies. It is worth noting the mapping $\pi$ is parametrized by the PRM nodes, i.e., $\mathcal{V}$. Thus, more rigorously, it can be written as $\pi(\cdot; \mathcal{V})$. For a given environment, there are infinite possible PRM graphs (and thus $\mathcal{V}$'s), any of which gives rise to a FIRM policy $\pi$. The set of all these possible FIRM policies are referred to as "admissible policies" and is denoted by $\Pi$.

The generic algorithms for offline construction of FIRM and online planning on FIRM are presented in Algorithms 1 and 2, respectively. The concrete instantiations of these algorithms for the Gaussian belief space are given in [1].

We would also like to quantify the quality of the solution $\pi$. To this end, we require the probability of success of the policy $\pi^g$ at the higher level Markov chain on $B_i$'s given by Eq.(4b). The DP in Eq.(4b) has $N_v + 1$ states $\{S_1, S_2, \cdots, S_{N_v+1}\}$ that can be decomposed into three disjoint classes: the goal class $S_1 = B_{goal}$, the failure class $S_2 = F$, and the transient class $\{S_3, S_4, \cdots, S_{N_v+1}\} = \{B_1, B_2, \cdots, B_{N_v}\} \setminus B_{goal}$. The goal and failure classes are recurrent classes of this Markov chain. As a result, the transition probability matrix of this higher level $N_v + 1$ state Markov chain can be decomposed as follows [18]:

**Algorithm 1:** Generic Construction of FIRM (Offline)

1. Construct a PRM with nodes $\mathcal{V} = \{\mathbf{v}_j\}$ and edges $\{e_{ij}\}$;
2. For each PRM node $\mathbf{v}_j$, design a controller $\mu^j$ and compute its corresponding reachable FIRM node $B_j$;
3. For each $B_i$ and $\mu^j \in \mathbb{M}(i)$, compute the cost, collision probabilities and transition probabilities associated with going from $B_i$ to $B_j$;
4. Solve the FIRM MDP to compute feedback $\pi^g$ over FIRM nodes, and compute the $\pi$ accordingly.

**Algorithm 2:** Generic planning on FIRM (Online)

1. Given an initial belief $b_0$, invoke the controller $\mu_0(\cdot)$ in Eq.(6), to absorb the robot into some FIRM node $B_i$;
2. Given the system is in set $B_i$, invoke the higher level feedback policy $\pi^g$ to choose the lower level feedback controller $\mu^j(\cdot) = \pi^g(B_i)$;
3. Let the node-controller $\mu^j(\cdot)$ execute until absorption into the $B_j$ or failure;
4. Repeat steps 2-3 until absorption into the goal node $B_{goal}$ or failure.

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_g & 0 & 0 \\ 0 & \mathcal{P}_f & 0 \\ \mathcal{R}_g & \mathcal{R}_f & \mathcal{Q} \end{bmatrix}. \tag{7}$$

The $(i, j)$-th component of $\mathcal{P}$ represents the transition probability from $S_j$ to $S_i$, i.e., $\mathcal{P}[i, j] = \mathbb{P}(S_i|S_j, \pi^g(S_j))$. Moreover $\mathcal{P}_g = \mathbb{P}(S_1|S_1, \pi^g(S_1)) = 1$ and $\mathcal{P}_f = \mathbb{P}(F|F, \cdot) = 1$, since goal and failure classes are recurrent classes, i.e., the system stops once it reaches the goal or it fails. $\mathcal{Q}$ is a matrix that represents the transition probabilities between belief nodes $B_i$ in transient class, whose $(i, j)$-th element is $\mathcal{Q}[i, j] = \mathbb{P}(S_{i+2}|S_{j+2}, \pi^g(S_{j+2}))$. Vectors $\mathcal{R}_g$ and $\mathcal{R}_f$ are $(N_v - 1) \times 1$ vectors that represent the probability of the transient nodes $\mathbb{V} \setminus B_{goal}$ getting absorbed into the goal node and the failure set, respectively, i.e., $\mathcal{R}_g[j] = \mathbb{P}(S_1|S_{j+2}, \pi^g(S_{j+2}))$ and $\mathcal{R}_f[j] = \mathbb{P}(S_2|S_{j+2}, \pi^g(S_{j+2}))$. Then, it can be shown that the success probability from any desired node $S_i \in \mathbb{V} \setminus B_{goal}$ is given by the $i$-th component of the vector $\mathcal{P}^s$ [18]:

$$\mathbb{P}(\text{success}|S_i, \pi) = \Gamma^T_{i-2}\mathcal{P}^s, \quad \mathcal{P}^s = (I - \mathcal{Q})^{-1}\mathcal{R}_g. \tag{8}$$

where $\Gamma_i$ is a column vector with all elements equal to zero but the $i$-th element, which is one.

Thus, given that we can suitably construct the local controllers $\mu^i(\cdot)$, the sets $B_i$, evaluate the transition costs $C^{\mu^i}(\cdot)$ and the transition probabilities $P^{\mu^i}(\cdot|\cdot)$, we can transform the intractable DP in Eq.(1) corresponding to the original POMDP into the solvable DP in Eq.(4) corresponding to the FIRM. A procedure for constructing the local controllers $\mu^i(\cdot)$, sets $B_i$, costs $C^{\mu^i}(\cdot)$ and probabilities $P^{\mu^i}(\cdot|\cdot)$ is presented in [1].

## III. Probabilistic Completeness Under Uncertainty

We start by reviewing the definition of success and probabilistic completeness in the deterministic case, and then we extend these definitions to the stochastic case.

*Success in the deterministic case*: In the deterministic case, such as conventional PRM, the outcome of the planning algorithm is a path. Thus, success is defined for paths: For a given initial and goal point, a successful path is a path connecting the start point to the goal point, which entirely lies in the obstacle-free space.

*Probabilistic completeness in the deterministic case*: In the absence of uncertainty, a sampling-based motion planning algorithm is probabilistically complete if by increasing the number of samples, the probability of finding a successful path, if one exists, asymptotically approaches to one.

*Difference between deterministic and probabilistic case*: In the presence of uncertainty, success cannot be defined for a path and it has to be defined for a policy. Indeed, on a given path, different policies may result in different success probabilities. Moreover, under uncertainty, one can only assign the probability for reaching goal. Thus, to define a success for a policy we consider a threshold $p_{min}$ and decide about success or failure accordingly.

*Successful policy*: In the presence of uncertainty, the solution of the planning algorithm is a policy (feedback) within the class of admissible policies. Therefore, success is defined for policies: For a given initial belief $b_0$ and goal region $B_g$, successful policy is a policy within the class of admissible policies under which the probability of reaching goal from the given initial point is greater than some predefined threshold $p_{min}$. In other words, $\pi \in \Pi$ is successful if $\mathbb{P}(\text{success}|b_0, \pi) := \mathbb{P}(B_g|b_0, \pi) > p_{min}$.

*Feasible initial belief*: A belief $b_0 \in \mathbb{B}$ is a feasible initial belief for class $\Pi$, if there exists a policy $\pi \in \Pi$ such that $\mathbb{P}(\text{success}|b_0, \pi) > p_{min}$. The set of all feasible initial beliefs corresponding to a class $\Pi$ is denoted by $\mathbb{B}_\Pi$. It is worth noting that the richer the set of admissible policies $\Pi$, the greater the set of feasible initial beliefs $\mathbb{B}_\Pi$. For example, in obstacle-free FIRM with stationary Linear Quadratic Gaussian (LQG) controllers as the local controllers, the set of all Gaussian beliefs is a subset of $\mathbb{B}_\Pi$ [1].

*Globally successful policy*: Instead of a single initial belief, we can also define the concept of successful policy for $\mathbb{B}_\Pi$. In other words, for a given goal region, a policy $\pi \in \Pi$ is globally successful, if the probability of reaching goal from any belief in $\mathbb{B}_\Pi$ is greater than $p_{min}$. In other words, $\pi \in \Pi$ is globally successful if $\mathbb{P}(\text{success}|b_0, \pi) = \mathbb{P}(B_g|b_0, \pi) > p_{min},\ \forall b_0 \in \mathbb{B}_\Pi$.

*Probabilistic completeness under uncertainty*: Probabilistic completeness can be defined based on either one of the definitions for the successful policy. Suppose there exists a (globally) successful policy $\pi \in \Pi$. Then, a sampling-based motion planning algorithm is probabilistically complete under uncertainty, if by increasing the number of samples without bound, the probability of finding a (globally) successful policy is one. In other words, if there exists a globally successful policy $\pi \in \Pi$, we have following property:

$$\lim_{N_v \to \infty} \mathbb{P}(B_g|b_0, \pi(\cdot; \mathcal{V})) > p_{min},\ \forall b_0 \in \mathbb{B}_\Pi, \qquad (9)$$

where $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^{N_v}$.

## IV. Probabilistic Completeness of FIRM

In this section, we present a result on the probabilistic completeness of the FIRM. The result and its proof is stated for the more general case of globally successful policies. It is worth noting that throughout this section by the word "continuous", we mean "Lipschitz continuous". Also, the norm $\|\cdot\|$ is the supremum norm, when it is applied to functions. If $\|\cdot\|$ is applied on operators, it stands for the operator norm [19].

Before stating the main proposition, we introduce some notation and assumptions that reflects the properties of FIRM in a more rigorous way.

*Assumption 1:*

- Local control laws are continuous functions of their parameters, i.e., for the $j$-th local controller, the mapping $\mu^j(\cdot; \mathbf{v}_j) : \mathbb{B} \to \mathbb{U}$ is a continuous function in the parameter $\mathbf{v}_j$.
- The transition pdf on h-state, i.e., $p(\mathcal{X}'|\mathcal{X}, u)$ is a continuous function of control $u$, i.e., there exists a $c_1 < \infty$, such that $\|p(\mathcal{X}'|\mathcal{X}, u) - p(\mathcal{X}'|\mathcal{X}, \check{u})\| \leq c_1\|u - \check{u}\|$.

Based on Assumption 1, the continuity of the transition probability induced by the local controllers in its parameter is deduced, i.e., we have the following result:

*Proposition 1: There exist $c_2 < \infty$ such that*

$$\|p(\mathcal{X}'|\mathcal{X}, \mu(b; \mathbf{v})) - p(\mathcal{X}'|\mathcal{X}, \check{\mu}(b; \check{\mathbf{v}}))\| \leq c_2\|\mathbf{v} - \check{\mathbf{v}}\|. \qquad (10)$$

Consider the h-state space $\mathbb{X}_h = \mathbb{X} \times \mathbb{B}$ that contains all possible h-states $\mathcal{X} = (X, b)$. The stopping regions in the belief space $\{B_i\}$ and the stopping region in the state space $F$, both can be extended to the h-state space, respectively denoted by $\{\mathcal{B}_i\}$ and $\mathcal{F}$, where $\mathcal{B}_i \subset \mathbb{X}_h$ and $\mathcal{F} \subset \mathbb{X}_h$.

$$\mathcal{B}_j = \{(X, b)| X \in \mathbb{X}_{free}, b \in B_j\}, \qquad (11)$$

$$\mathcal{F} = \{(X, b)| X \in F, b \in \mathbb{B}\}, \qquad (12)$$

$$\mathcal{S}_j = \mathcal{B}_j \cup \mathcal{F}, \qquad (13)$$

where, $\mathcal{S}_j$ denotes the entire stopping region under the local controller $\mu^j$. Based on these definitions, rephrasing the condition in FIRM for designing the node regions, we get the following assumption:

*Assumption 2:* We assume that after $N < \infty$ steps, the probability of getting absorbed into the stopping region $\mathcal{S}_j$ corresponding to $\mathbf{v}_j$ is greater than zero under $\mu^j$, i.e., we assume for each $\mu^j$, $\inf_{\mathcal{X}} \mathbb{P}_n(\mathcal{S}_j|\mathcal{X},\mu^j) = \beta > 0$, for all $n > N$.
In [1], it is shown how such a regions can be designed when the local controllers $\mu$ are LQG controllers.

Finally, we state the following assumption, in which we emphasize the fact that as $\mathbf{v} \to \check{\mathbf{v}}$, the probability measure induced by the local controller $\mu(\cdot;\mathbf{v})$ over the sets $\mathcal{B}$ and $\check{\mathcal{B}}$ have to converge also. This assumption almost always holds in practical cases.

*Assumption 3:* Consider the controllers $\mu(\cdot;\mathbf{v})$, and $\check{\mu}(\cdot;\check{\mathbf{v}})$, whose corresponding extended absorption regions are denoted by $\mathcal{B}$ and $\check{\mathcal{B}}$, respectively. We assume that there exist an $r > 0$ and $c' < \infty$, such that for $\|\mathbf{v}-\check{\mathbf{v}}\| \leq r$, we have:

$$\|\mathbb{P}_1(\mathcal{B}\ominus\check{\mathcal{B}}|\mathcal{X},\mu)\| \leq c'\|\mathbf{v}-\check{\mathbf{v}}\| \tag{14}$$

where, $\ominus$ is the symmetric difference operator, i.e., $\mathcal{B}\ominus\check{\mathcal{B}} = (\mathcal{B}-\check{\mathcal{B}})\cup(\check{\mathcal{B}}-\mathcal{B})$.

Now, we are ready to state the main proposition:

*Proposition 2: Given Assumptions 1, 2, and 3, FIRM is probabilistically complete under uncertainty.*

*Proof:* We prove this proposition in three steps:

1) *Equivalence of completeness and continuity*: In the first step, we show that FIRM is probabilistically complete under uncertainty, if the probability of success under the policy $\pi \in \Pi$, i.e., $\mathbb{P}(\text{success}|\pi, b_0)$ is a continuous function of the underlying PRM nodes, i.e., $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^{N_v}$, for all $b_0 \in \mathbb{B}_\Pi$.
2) *Continuity of success probability*: We show that $\mathbb{P}(\text{success}|\pi, b_0)$ is continuous wrt $\mathcal{V}$, if the absorption probabilities $\mathbb{P}(B_i|\mu^i, b)$ is continuous wrt $\mathbf{v}_i$, for all $i$ and $b$.
3) *Continuity of absorption probabilities*: We show that the absorption probability $\mathbb{P}(B_i|\mu^i, b)$ is continuous wrt $\mathbf{v}_i$, for all $i$ and $b$, given the transition probability $p(\mathcal{X}'|\mathcal{X},\mu^i(b))$ is continuous wrt $\mathbf{v}_i$, i.e., given Eq.(10).

In the following subsections, we prove above steps, in the mentioned order.

### A. Equivalence of completeness and continuity

Based on the definition of probabilistic completeness under uncertainty, if there exist a globally successful policy $\check{\pi}$, FIRM has to find a globally successful policy $\pi$ as the number of FIRM nodes increases unboundedly. Thus, we start by assuming that there exists a globally successful policy $\check{\pi} \in \Pi$. Since each policy in $\Pi$ is parametrized by a PRM graph, there exists a PRM with nodes $\check{\mathcal{V}} = \{\check{\mathbf{v}}_i\}$ that parametrizes the policy $\check{\pi}$.

In the state space, for every node $\check{\mathbf{v}}_i$, consider a ball $\check{\Omega}_i$ with radius $\delta > 0$, centred at $\check{\mathbf{v}}_i$. Now, suppose there exists a PRM, with the set of nodes $\mathcal{V} = \{\mathbf{v}_i\}$, where $\mathbf{v}_i \in \check{\Omega}_i$, i.e. $\|\mathbf{v}_i-\check{\mathbf{v}}_i\| \leq \delta$ for all $i$. Let us denote the policy parametrized by $\mathcal{V}$ as $\pi$. Since $\check{\pi}$ is a globally successful policy, we know $\mathbb{P}(\text{success}|\check{\pi}, b_0) > p_{min}$ for all $b_0 \in \mathbb{B}_\Pi$. Thus, we can define $\epsilon = \mathbb{P}(\text{success}|\check{\pi}, b_0) - p_{min} > 0$. If the probability of success $\mathbb{P}(\text{success}|\check{\pi}, b_0)$ is a continuous function of the underlying PRM nodes $\mathcal{V}' = \{\check{\mathbf{v}}_i\}$, then there exist a $\delta$, for which we have $|\mathbb{P}(\text{success}|\check{\pi}, b_0) - \mathbb{P}(\text{success}|\pi, b_0)| < \epsilon$, and thus, $\mathbb{P}(\text{success}|\pi, b_0) > p_{min}$. Therefore, for sufficiently small $\delta > 0$ the obtained policy $\pi$ is a successful policy.

Since $\delta > 0$, the regions $\check{\Omega}_i$ are the sets with strictly positive probability measures under the sampling algorithm of PRM, e.g., uniform sampling. Thus, starting with any PRM, if we increase the number of nodes, a PRM node will eventually be chosen at every $\check{\Omega}_i$, with probability one. Therefore the policy constructed based on these nodes will have a success probability greater than $p_{min}$. Thus, FIRM is probabilistically complete.

### B. Continuity of success probability

Given that $\mathbb{P}(B_i|\mu^i, b)$ is continuous wrt $\mathbf{v}_i$, for all $i$, we want to show that $\mathbb{P}(\text{success}|\pi, b_0)$ is continuous wrt all $\mathbf{v}_i$. First, let us look at the structure of the success probability.

$$\mathbb{P}(\text{success}|\pi, b_0) = \mathbb{P}(B(\mu_0)|\mu_0, b_0)\mathbb{P}(\text{success}|\pi, B(\mu_0)), \tag{15}$$

where, $\mu_0$ is computed using Eq.(6). The term $\mathbb{P}(B(\mu_0)|\mu_0, b_0)$ in the right hand side of Eq.(15) is continuous because the continuity of $\mathbb{P}(B_i|\mu^i, b)$ for all $i$ is assumed in this subsection. Thus, we only need to show the continuity of the second term in Eq.(15). Without loss of generality we can consider $B_i = B(\mu_0)$. Then, it is desired to show that $\mathbb{P}(\text{success}|\pi, B_i)$ is continuous wrt $\mathbf{v}_i$ for all $i$.

As we saw in Section II-B, the probability of success from the $i$-th FIRM node is as follows:

$$\mathbb{P}(\text{success}|\pi, B_i) = \Gamma_i^T(I-\mathcal{Q})^{-1}\mathcal{R}_g, \tag{16}$$

Moreover, we can consider $B_g = B_N$ without loss of generality; then, the $(i,j)$-th element of matrix $\mathcal{Q}$ is $\mathcal{Q}[i,j] = \mathbb{P}(B_i|B_j, \pi^g(B_j))$, and the $j$-th element of vector $\mathcal{R}_g$ is $\mathcal{R}_g[j] = \mathbb{P}(B_N|B_j, \pi^g(B_j))$. Since we considered the $B_j$ as the stopping region of the local controller $\mu^j$, we have:

$$\mathbb{P}(B_j|B_i, \mu^l) = 0, \text{ if } l \neq j. \tag{17}$$

Therefore, all the non-zero elements in the matrices $\mathcal{R}_g$ and $\mathcal{Q}$ are of the form $\mathbb{P}(B_j|B_i, \mu^j)$. Thus, Given the continuity of $\mathbb{P}(B_j|b, \mu^j)$, the transition probability $\mathbb{P}(B_j|B_i, \mu^j)$ are continuous and the matrices $\mathcal{R}_g$ and $\mathcal{Q}$ are continuous. Therefore, $\mathbb{P}(\text{success}|\pi, B_i)$ and thus $\mathbb{P}(\text{success}|\pi, b_0)$ are continuous wrt underlying PRM nodes.

### C. Continuity of absorption probabilities

The absorption probability into the FIRM nodes are computed through solving following integral equation that comes from the law of total probability:

$$\mathbb{P}(\mathcal{B}_j|\mathcal{X},\mu^j) = \int_{\mathbb{X}_h} p^{\mu^j}(\mathcal{X}'|\mathcal{X})\mathbb{P}(\mathcal{B}_j|\mathcal{X}',\mu^j)d\mathcal{X}', \tag{18}$$

with the conditions:

$$\mathbb{P}(\mathcal{B}_j|\mathcal{X},\mu^j) = \begin{cases} 1, & \text{if } \mathcal{X} \in \mathcal{B}_j \\ 0, & \text{if } \mathcal{X} \in \mathcal{F} \end{cases}. \tag{19}$$

Plugging the condition equations into the Eq.(18), we can write Eq.(18) as follows:

$$\mathbb{P}(\mathcal{B}_j|\mathcal{X},\mu^j) = \int_{\mathcal{B}_j} p^{\mu^j}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' + \int_{\overline{\mathcal{S}_j}} p^{\mu^j}(\mathcal{X}'|\mathcal{X})\mathbb{P}(\mathcal{B}_j|\mathcal{X}',\mu^j)d\mathcal{X}'. \quad (20)$$

Henceforth, we drop the index $j$ to unclutter the expressions, and the coming results hold for any $j$. Thus, we can write:

$$\begin{aligned}\mathbb{P}(\mathcal{B}|\mathcal{X},\mu) &= \int_{\mathcal{B}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' + \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\mathbb{P}(\mathcal{B}|\mathcal{X}',\mu)d\mathcal{X}' \\ &= R(\mathcal{X}) + \mathbf{T}_{\mathcal{S}}\left[\mathbb{P}(\mathcal{B}|\cdot,\mu)\right](\mathcal{X}), \quad (21)\end{aligned}$$

where, the operator $\mathbf{T}_{\mathcal{S}}$ and the function $R(\mathcal{X})$ are defined as:

$$\mathbf{T}_{\mathcal{S}}\left[f(\cdot)\right](\mathcal{X}) := \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})f(\mathcal{X}')d\mathcal{X}', \quad (22)$$

$$R(\mathcal{X}) := \int_{\mathcal{B}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'. \quad (23)$$

The equation in Eq.(21) is an inhomogeneous Fredholm equation of second type. The solution of this equation can be expressed as a Liouville-Neumann series [19]:

$$\mathbb{P}(\mathcal{B}|\mathcal{X},\mu) = \sum_{n=1}^{\infty} \mathbf{T}_{\mathcal{S}}^n\left[R(\cdot)\right](\mathcal{X}). \quad (24)$$

Before proceeding with proof of continuity of absorption probability, we state the following lemma on the operator $\mathbf{T}_{\mathcal{S}}$, whose proof is given in Appendix 1.

*Lemma 1: According to the Assumption 2, we have:*

$$\begin{cases} \|\mathbf{T}_{\mathcal{S}}^n\| \le 1, & n < N \\ \|\mathbf{T}_{\mathcal{S}}^n\| \le \alpha < 1, & n \ge N \\ \sum_{n=0}^{\infty}\|\mathbf{T}_{\mathcal{S}}^n\| \le c < \infty. \end{cases} \quad (25)$$

Note that since here $\|\cdot\|$ acts on the operators, it stands for the operator norm [19].

*Corollary 1: Series $\sum_{n=0}^{\infty}\mathbf{T}_{\mathcal{S}}^n[R]$ is a convergent series, and therefore, we can define the resolvent operator $(I-\mathbf{T}_{\mathcal{S}})^{-1}[R] = \sum_{n=0}^{\infty}\mathbf{T}_{\mathcal{S}}^n[R]$, where $\|(I-\mathbf{T}_{\mathcal{S}})^{-1}\| \le c < \infty$.*

According to the Corollary 1, the success probability can be written using the defined resolvent operators as:

$$\mathbb{P}(\mathcal{B}|\mathcal{X},\mu) = (I-\mathbf{T}_{\mathcal{S}})^{-1}[R(\cdot)](\mathcal{X}). \quad (26)$$

To show $\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)$ is continuous wrt $\mathbf{v}$, we perturb $\mathbf{v}$ to some $\check{\mathbf{v}}$, such that $\|\mathbf{v}-\check{\mathbf{v}}\| < r$. The local controller associated with node $\check{\mathbf{v}}$ is referred to as $\check{\mu}$, whose successful absorption region is denoted by $\check{\mathcal{B}}$ and stopping region is $\check{\mathcal{S}}$. Similarly the corresponding transient operator and recurrent function are referred to as $\check{\mathbf{T}}_{\check{\mathcal{S}}}$ and $\check{R}$. Finally, the success probability associated with the perturbed node $\check{\mathbf{v}}$ is $\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu})$. To shorten the statements, we refer to $\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)$ and $\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu})$ respectively by $\mathfrak{P}(\mathcal{X})$ and $\check{\mathfrak{P}}(\mathcal{X})$. As a result of node perturbation, the success probability is perturbed as:

$$\begin{aligned}&\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)-\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu}) := \mathfrak{P}-\check{\mathfrak{P}} = R+\mathbf{T}_{\mathcal{S}}[\mathfrak{P}]-\check{R}-\check{\mathbf{T}}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}] \\ &= R-\check{R}+\mathbf{T}_{\mathcal{S}}[\mathfrak{P}]-\mathbf{T}_{\mathcal{S}}[\check{\mathfrak{P}}]+\mathbf{T}_{\mathcal{S}}[\check{\mathfrak{P}}]-\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]+\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]-\check{\mathbf{T}}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}] \\ &= (R-\check{R})+\mathbf{T}_{\mathcal{S}}[\mathfrak{P}-\check{\mathfrak{P}}]+(\mathbf{T}_{\mathcal{S}}-\mathbf{T}_{\check{\mathcal{S}}})[\check{\mathfrak{P}}]+(\mathbf{T}_{\check{\mathcal{S}}}-\check{\mathbf{T}}_{\check{\mathcal{S}}})[\check{\mathfrak{P}}] \quad (27)\end{aligned}$$

where

$$\mathbf{T}_{\check{\mathcal{S}}}\left[f(\cdot)\right](\mathcal{X}) := \int_{\overline{\check{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})f(\mathcal{X}')d\mathcal{X}'. \quad (28)$$

Let us define the operators $\mathbf{T}_{\Delta\mathcal{S}} := (\mathbf{T}_{\mathcal{S}}-\mathbf{T}_{\check{\mathcal{S}}})$ and $\Delta\mathbf{T}_{\check{\mathcal{S}}} := (\mathbf{T}_{\check{\mathcal{S}}}-\check{\mathbf{T}}_{\check{\mathcal{S}}})$. Now, based on Corollary 1, we can write:

$$\mathfrak{P}-\check{\mathfrak{P}} = (I-\mathbf{T}_{\mathcal{S}})^{-1}\left[R-\check{R}+\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]+\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\right], \quad (29)$$

and thus following inequality holds on the supremum norm of the perturbation of the absorption probability:

$$\begin{aligned}&\|\mathfrak{P}-\check{\mathfrak{P}}\| \\ &\le \|(I-\mathbf{T}_{\mathcal{S}})^{-1}\|\left(\|R-\check{R}\|+\|\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]\|+\|\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|\right) \\ &\le c\left(\|R-\check{R}\|+\|\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]\|+\|\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|\right) \\ &= c\left(\|K_1(\mathcal{X})\|+\|K_2(\mathcal{X})\|+\|K_3(\mathcal{X})\|\right), \quad (30)\end{aligned}$$

where, $K_1(\mathcal{X}) := R(\mathcal{X})-\check{R}(\mathcal{X})$, $K_2(\mathcal{X}) := \mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}(\cdot)](\mathcal{X})$, and $K_3(\mathcal{X}) := \Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}(\cdot)](\mathcal{X})$. In the following we bound $K_1$, $K_2$, and $K_3$, and thus bound the $\|\mathfrak{P}-\check{\mathfrak{P}}\|$, accordingly. In this process we will use the result, stated in the following lemma.

*Lemma 2: Consider the bounded function $0 \le f(\mathcal{X}) \le 1$, and kernel $k(\mathcal{X}',\mathcal{X}) \ge 0$. Then, for any set $\mathcal{A}$, we have:*

$$\|\int_{\mathcal{A}} k(\mathcal{X}',\mathcal{X})f(\mathcal{X}')d\mathcal{X}'\| \le \|\int_{\mathcal{A}} k(\mathcal{X}',\mathcal{X})d\mathcal{X}'\| \quad (31)$$

*Proof:* Given the properties of $f(\cdot)$ and $k(\cdot,\cdot)$, we have $k(\mathcal{X}',\mathcal{X})f(\mathcal{X}') \le k(\mathcal{X}',\mathcal{X})$, for all $\mathcal{X}$ and $\mathcal{X}'$. Thus, the stated result follows from taking integral from both sides with respect to $\mathcal{X}'$ and then taking supremum norm with respect to $\mathcal{X}$. ■

*1) Bound for $K_1(\mathcal{X})$:* The supremum norm of $K_1(\mathcal{X})$ is:

$$\begin{aligned}&\|K_1(\mathcal{X})\| = \|R(\mathcal{X})-\check{R}(\mathcal{X})\| \\ &\le \|\int_{\mathcal{B}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' - \int_{\check{\mathcal{B}}} p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\| \\ &= \|\int_{\mathcal{B}\cap\check{\mathcal{B}}}[p^{\mu}(\mathcal{X}'|\mathcal{X})-p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})]d\mathcal{X}' \\ &+ \int_{\mathcal{B}-\check{\mathcal{B}}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' - \int_{\check{\mathcal{B}}-\mathcal{B}} p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\| \\ &\le \int_{\mathcal{B}\cap\check{\mathcal{B}}}\|p^{\mu}(\mathcal{X}'|\mathcal{X})-p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})\|d\mathcal{X}' \\ &+ \|\int_{\mathcal{B}-\check{\mathcal{B}}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' + \int_{\check{\mathcal{B}}-\mathcal{B}} p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\| \\ &\overset{\text{from (10)}}{\le} \int_{\mathcal{B}\cap\check{\mathcal{B}}} c_2\|\mathbf{v}-\check{\mathbf{v}}\|d\mathcal{X}' + \|\mathbb{P}_1(\mathcal{B}\ominus\check{\mathcal{B}}|\mathcal{X},\mu)\|\end{aligned}$$

$$+ \|\mathbb{P}_1(\breve{\mathcal{B}} \ominus \mathcal{B}|\mathcal{X}, \breve{\mu})\|$$
$$\overset{\text{from (14)}}{\leq} c_2'\|\mathbf{v} - \check{\mathbf{v}}\| + 2c'\|\mathbf{v} - \check{\mathbf{v}}\| = c_3\|\mathbf{v} - \check{\mathbf{v}}\|, \quad (32)$$

where, $c_2' < \infty$ and $c_3 = c_2' + 2c' < \infty$. In the penultimate inequality, we also used the fact that $\mathbb{P}_1(\breve{\mathcal{B}} - \mathcal{B}|\mathcal{X}, \breve{\mu}) \leq \mathbb{P}_1(\breve{\mathcal{B}} \ominus \mathcal{B}|\mathcal{X}, \breve{\mu})$ and $\mathbb{P}_1(\mathcal{B} - \breve{\mathcal{B}}|\mathcal{X}, \mu) \leq \mathbb{P}_1(\mathcal{B} \ominus \breve{\mathcal{B}}|\mathcal{X}, \mu)$ because $\breve{\mathcal{B}} - \mathcal{B} \subseteq \breve{\mathcal{B}} \ominus \mathcal{B}$ and $\mathcal{B} - \breve{\mathcal{B}} \subseteq \mathcal{B} \ominus \breve{\mathcal{B}}$.

*2) Bound for $K_2(\mathcal{X})$:* We have:

$$\|K_2(\mathcal{X})\| = \|\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]\| = \|\mathbf{T}_{\mathcal{S}}[\check{\mathfrak{P}}] - \mathbf{T}_{\breve{\mathcal{S}}}[\check{\mathfrak{P}}]\|$$
$$\leq \| \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}' - \int_{\overline{\breve{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|$$
$$= \| \int_{\overline{\mathcal{S}}-\overline{\breve{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}' - \int_{\overline{\breve{\mathcal{S}}}-\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|$$
$$\leq \| \int_{\overline{\mathcal{S}}-\overline{\breve{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}' + \int_{\overline{\breve{\mathcal{S}}}-\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|$$
$$= \| \int_{\overline{\mathcal{S}}\ominus\overline{\breve{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\| \overset{\text{from (31)}}{\leq} \| \int_{\overline{\mathcal{S}}\ominus\overline{\breve{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\|$$
$$= \|\mathbb{P}_1(\overline{\mathcal{S}} \ominus \overline{\breve{\mathcal{S}}}|\mathcal{X}, \mu)\| \leq \|\mathbb{P}_1(\overline{\mathcal{B}} \ominus \overline{\breve{\mathcal{B}}}|\mathcal{X}, \mu)\| \quad (33)$$
$$= \|\mathbb{P}_1(\mathcal{B} \ominus \breve{\mathcal{B}}|\mathcal{X}, \mu)\| \overset{\text{from (14)}}{\leq} c'\|\mathbf{v} - \check{\mathbf{v}}\|.$$

The penultimate inequality and equality follow from the relations $\overline{\mathcal{S}} \ominus \overline{\mathcal{S}'} \subseteq \overline{\mathcal{B}} \ominus \overline{\mathcal{B}'}$ and $\overline{\mathcal{B}} \ominus \overline{\mathcal{B}'} = \mathcal{B} \ominus \mathcal{B}'$, respectively.

*3) Bound for $K_3(\mathcal{X})$:* We have:

$$\|K_3(\mathcal{X})\| = \|\Delta\mathbf{T}_{\breve{\mathcal{S}}}[\check{\mathfrak{P}}]\| = \|\mathbf{T}_{\breve{\mathcal{S}}}[\check{\mathfrak{P}}] - \check{\mathbf{T}}_{\breve{\mathcal{S}}}[\check{\mathfrak{P}}]\|$$
$$\leq \| \int_{\overline{\breve{\mathcal{S}}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}' - \int_{\overline{\breve{\mathcal{S}}}} p^{\breve{\mu}}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|$$
$$= \| \int_{\overline{\breve{\mathcal{S}}}} \left(p^{\mu}(\mathcal{X}'|\mathcal{X}) - p^{\breve{\mu}}(\mathcal{X}'|\mathcal{X})\right)\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|$$
$$\leq \int_{\overline{\breve{\mathcal{S}}}} \|p^{\mu}(\mathcal{X}'|\mathcal{X}) - p^{\breve{\mu}}(\mathcal{X}'|\mathcal{X})\|\|\check{\mathfrak{P}}(\mathcal{X}')\|d\mathcal{X}'$$
$$\overset{\text{from (10)}}{\leq} \int_{\overline{\breve{\mathcal{S}}}} c_2\|\mathbf{v} - \check{\mathbf{v}}\|d\mathcal{X}' = c_2''\|\mathbf{v} - \check{\mathbf{v}}\|. \quad (34)$$

where, $c_2'' < \infty$.

Therefore, based on Eq.(32), Eq.(33), Eq.(34), and Eq.(30), we can conclude that:

$$\|\mathbb{P}(\mathcal{B}|\mathcal{X}, \mu) - \mathbb{P}(\breve{\mathcal{B}}|\mathcal{X}, \breve{\mu})\| \leq c_4\|\mathbf{v} - \check{\mathbf{v}}\|, \quad (35)$$

where $c_4 = c_3 + c' + c_2'' < \infty$, which completes the proof that absorption probability under the controller $\mu$ is continuous in the PRM node $\mathbf{v}$.

Therefore, based on steps two and three of the proof, we conclude the success probability is a continuous function of PRM nodes, i.e., for any $\epsilon > 0$, there exists a $\delta > 0$, such that if $\|\mathcal{V} - \check{\mathcal{V}}\| < \delta$, then $\|\mathbb{P}(\text{success}|b_0, \pi(\cdot; \mathcal{V})) - \mathbb{P}(\text{success}|b_0, \breve{\pi}(\cdot; \check{\mathcal{V}}))\| < \epsilon$. Thus, based on the step one of the proof, the probabilistic completeness of the FIRM is concluded.

■

## V. Discussion

The basic idea in probabilistic completeness under uncertainty stems from an idea similar to the one in path isolation-based analysis in planners for deterministic systems. Roughly speaking, in the path isolation argument for the sampling-based planners in the absence of uncertainty, if there is a successful path and a non-zero neighbourhood of this path, in which every path is successful, we can eventually find a path in this neighbourhood, by increasing the number of samples, unboundedly. Similarly, in the presence of uncertainty, if there is a successful policy, it is parametrized by some parameters. Thus, if there is exists a non-zero measure neighbourhood of these parameters in the parameter space, such that all parameters chosen in this neighbourhood leads to a successful policy, we can eventually reach a successful policy, by increasing the number of samples unboundedly and falling into the target neighbourhood.

In FIRM or similar approaches, such as Generalized Probabilistic Roadmap Methods (GPRM) [10], policy $\pi$ is parametrized by an underlying PRM graph nodes. Therefore, increasing the number of nodes in the underlying PRM, we can reach parameters arbitrarily close the parameters of a successful policy, if one exists, and thus based on continuity of the success probability in the parameters, we can eventually get a successful policy, by increasing the number of PRM nodes, unboundedly.

## VI. Conclusion

In this paper, we reformulated the sampling-based robot motion planning problem under uncertainty, in terms of the local controllers and their interactions. Inspired by the concept of the probabilistic completeness in the deterministic situation, we introduced the concept of probabilistic completeness under uncertainty. Accordingly, we proposed a way to approach proving the probabilistic completeness of the motion planning methods under uncertainty. Finally, we showed that the FIRM algorithm for motion planning under uncertainty is a probabilistically complete algorithm.

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## APPENDIX I
## PROOF OF LEMMA 1

*Proof:* Let us denote the $n$-th iterated kernel of $\mathbf{T}_{\mathcal{S}}$ as $p_n(\mathcal{X}'|\mathcal{X},\mu)$. Since this iterated kernel is a pdf, we have $p_n(\mathcal{X}'|\mathcal{X},\mu) \geq 0, \quad \forall\mathcal{X},\forall\mathcal{X}',\forall n$. If we denote the domain of operator $\mathbf{T}_{\mathcal{S}}$ by $\mathcal{D}$, we know that for all $f \in \mathcal{D}$, we have $0 \leq f(\mathcal{X}) \leq 1$, because $f(\mathcal{X})$ is the probability of some given set $\mathcal{S}$ under some given controller invoked at point $\mathcal{X}$. Thus, it cannot be negative or greater than one and based on lemma 2, we have:

$$\|\int_{\overline{\mathcal{S}}} p_n(\mathcal{X}'|\mathcal{X},\mu)f(\mathcal{X}')d\mathcal{X}'\| \leq \|\int_{\overline{\mathcal{S}}} p_n(\mathcal{X}'|\mathcal{X},\mu)d\mathcal{X}'\|$$
$$= \|\mathbb{P}_n(\overline{\mathcal{S}}|\mathcal{X},\mu)\| = \|1-\mathbb{P}_n(\mathcal{S}|\mathcal{X},\mu)\| \tag{36}$$

Thus, we have

$$\|\mathbf{T}_{\mathcal{S}}[f]\| = \|\int_{\overline{\mathcal{S}}} p(\mathcal{X}'|\mathcal{X},\mu)f(\mathcal{X}')d\mathcal{X}'\| \leq \|\mathbb{P}(\overline{\mathcal{S}}|\mathcal{X},\mu)\| \leq 1$$

Based on the definition of operator norm, we have:

$$\|\mathbf{T}_{\mathcal{S}}\| = \sup_{f(\cdot)}\{\|\mathbf{T}_{\mathcal{S}}[f]\| \ : \ \forall f \in \mathcal{D}, \|f\| \leq 1\} \leq 1 \tag{37}$$

According to Assumption 2, there exists a finite number $N$, such that:

$$\inf_{\mathcal{X}} \mathbb{P}_n(\mathcal{S}|\mathcal{X},\mu) = \beta > 0, \quad \forall n > N \tag{38}$$

Thus, we have

$$\|\mathbb{P}_n(\overline{\mathcal{S}}|\mathcal{X},\mu)\| = 1-\beta < 1, \quad \forall n > N \tag{39}$$

Therefore, we can write:

$$\|\mathbf{T}^N_{\mathcal{S}}[f]\| = \|\int_{\overline{\mathcal{S}}} p_N(\mathcal{X}'|\mathcal{X},\mu)f(\mathcal{X}')d\mathcal{X}'\|$$
$$\leq \|\mathbb{P}_N(\overline{\mathcal{S}}|\mathcal{X},\mu)\| \leq \alpha < 1 \tag{40}$$

where $\alpha = 1-\beta$, and similar to Eq.(37), we get $\|\mathbf{T}^N_{\mathcal{S}}\| \leq \alpha < 1$. Thus, we also have:

$$\|\mathbf{T}^{N+1}_{\mathcal{S}}\| \leq \|\mathbf{T}^N_{\mathcal{S}}\|\|\mathbf{T}_{\mathcal{S}}\| \leq \alpha < 1$$

and similarly for all $n \geq N$, we have:

$$\|\mathbf{T}^n_{\mathcal{S}}\| \leq \alpha < 1, \quad \forall n \geq N$$

Now, consider the series: $\sum_{i=1}^{\infty}\|\mathbf{T}^n_{\mathcal{S}}\|$. We can split the sum to smaller pieces as follows:

$$\sum_{n=1}^{\infty}\|\mathbf{T}^n_{\mathcal{S}}\| = \sum_{n=1}^{N}\|\mathbf{T}^n_{\mathcal{S}}\| + \sum_{i=1}^{\infty}\sum_{n=iN+1}^{(i+1)N}\|\mathbf{T}^n_{\mathcal{S}}\|$$

but because $\|\mathbf{T}^{n+1}_{\mathcal{S}}\| \leq \|\mathbf{T}^n_{\mathcal{S}}\|$ for all $n \geq N$, we have

$$\sum_{n=iN+1}^{(i+1)N}\|\mathbf{T}^n_{\mathcal{S}}\| \leq N\|\mathbf{T}^{iN}_{\mathcal{S}}\|$$

Also, we know

$$\|\mathbf{T}^{iN}_{\mathcal{S}}\| \leq \|\mathbf{T}^N_{\mathcal{S}}\|^i \leq \alpha^i$$

and thus, we have:

$$\sum_{n=1}^{\infty}\|\mathbf{T}^n_{\mathcal{S}}\| = \underbrace{\sum_{n=1}^{N}\|\mathbf{T}^n_{\mathcal{S}}\|}_{\leq N-1+\alpha} + \sum_{i=1}^{\infty}\sum_{n=iN+1}^{(i+1)N}\|\mathbf{T}^n_{\mathcal{S}}\|$$
$$\leq N-1+\alpha+\sum_{i=1}^{\infty}N\alpha^i$$
$$= N-1+\alpha+\frac{N}{1-\alpha} = c < \infty \tag{41}$$

■

## APPENDIX II
## PROOF OF COROLLARY 1

*Proof:* We have

$$\|\sum_{n=0}^{\infty}\mathbf{T}^n_{\mathcal{S}}[R]\| \leq \sum_{n=0}^{\infty}\|\mathbf{T}^n_{\mathcal{S}}\|\|R\| \leq \sum_{n=0}^{\infty}\|\mathbf{T}^n_{\mathcal{S}}\| \leq c < \infty \tag{42}$$

Thus, series $\sum_{n=0}^{\infty}\mathbf{T}^n_{\mathcal{S}}[R]$ is a convergent series and we can define the operator $(I-\mathbf{T}_{\mathcal{S}})^{-1}[R] = \sum_{n=0}^{\infty}\mathbf{T}^n_{\mathcal{S}}[R]$. We have

$$\|(I-\mathbf{T}_{\mathcal{S}})^{-1}\| = \|\sum_{n=0}^{\infty}\mathbf{T}^n_{\mathcal{S}}\| \leq c < \infty \tag{43}$$

.

■